## Part III

## **Basics of quantum transport**

# Chapter 11 Preliminary Concepts

## 11.1 Two-Dimensional Electron Gas

An important system where quantum effects were observed is two-dimensional electron gas (2DEG). There are two basic systems where 2DEG has been studied. One of them is Si MOSFETs (metal-oxide-semiconductor field-effect transistors). A very good review of such systems is given in Ref. [7]. A typical device is shown in Fig. 11.1. A (100)Si surface serves



Figure 11.1: Band diagram showing conductance band  $E_C$ , valence band  $E_V$  and quasi-Fermi level  $E_F$ . A 2DEG is formed at the interface between the oxide (SiO<sub>2</sub>) and *p*-type silicon substrate as a consequence of the gate voltage  $V_g$ .

as a substrate while  $SiO_2$  layer behaves as an insulator. 2DEG is induced electrostatically by application a positive voltage  $V_g$ . The sheet density of 2DEG can be described as

$$n_s = \frac{\epsilon_{\rm ox}}{ed_{\rm ox}} (V_g - V_t)$$

where  $V_t$  is the threshold voltage for the barrier's creation

Another important systems with 2DEG involve modulation-doped GaAs-AlGaAs heterostructures. The bandgap in AlGaAs is wider than in GaAs. By variation of doping it is possible to move the Fermi level inside the forbidden gap. When the materials are put together, a unified level of chemical potential is established, and an inversion layer is formed at the interface.



Figure 11.2: Band structure of the interface between n-AlGa As and intrinsic GaAs, (a) before and (b) after the charge transfer.

The 2DEG created by a modulation doping can be squeezed into narrow channels by selective depletion in spatially separated regions. The simplest lateral confinement technique is to create split metallic gates in a way shown in Fig. 11.3 A typical nanostructure is shown in Fig. 11.4.

## 11.2 Basic Properties of Low-Dimensional Systems

### Wave Functions

Let us direct z-axis perpendicular to the plane of 2DEG. The wave function can be decoupled as

$$\Psi(\mathbf{r}, z) = \chi(z)\,\psi(\mathbf{r})$$

where  $\mathbf{r}$  is the vector in plane of 2DEG. Throughout our considerations we will assume that all the distances are much larger than interatomic distance and thus we will use the effective

![](_page_4_Figure_1.jpeg)

Figure 11.3: On the formation of a narrow channel by a split gate.

![](_page_4_Picture_3.jpeg)

Figure 11.4: Scanning electron microphotographs of nanostructures in GaAs-AlGaAs heterostructures. Taken from M. L. Roukes *et al.*, Phys. Rev. Lett. **59**, 3011 (1987).

mass approximation. A good approximation for the confining potential is a triangular one,

$$U(z) = \begin{cases} \infty & \text{at} \quad z < 0; \\ Fz & \text{at} \quad z > 0. \end{cases}$$

Then one can write the Schrödinger equation for the wave function  $\chi(z)$  as

$$\frac{\partial^2 \chi}{\partial z^2} + \frac{2m}{\hbar^2} (E - Fz)\chi = 0. \qquad (11.1)$$

Instead z we introduce a dimensionless variable

$$\zeta = \left(z - \frac{E}{F}\right) \left(\frac{2mF}{\hbar^2}\right)^{1/3}$$

The quantity

$$\ell_F = \left(\frac{2mF}{\hbar^2}\right)^{-1/3}$$

plays the role of characteristic localization length in z direction. Then Eq. (11.1) acquires the form

$$\chi'' - \zeta \chi = 0$$

which should be solved with the boundary conditions of finiteness at infinity and zero at z = 0. Such a solution has the form

$$\chi(\zeta) = A\operatorname{Ai}(\zeta).$$

Here  $\operatorname{Ai}(\zeta)$  is the Airy function defined as

$$\operatorname{Ai}(\zeta) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos(u^3/3 + u\zeta) \, du \, .$$

For large positive  $\zeta$  it decays exponentially,

$$\operatorname{Ai}(\zeta) \approx \frac{1}{2\zeta^{1/4}} e^{-(2/3)\zeta^{3/2}},$$

while for large negative *zeta* is is oscillatory,

$$\operatorname{Ai}(\zeta) \approx \frac{1}{|\zeta|^{1/4}} \sin\left(\frac{2}{3}|\zeta|^{3/2} + \frac{\pi}{4}\right) \,.$$

The energy spectrum E is defined by the roots  $\zeta_n$  of the equation

$$\operatorname{Ai}(\zeta) = 0, \quad \to \quad E_n = -E_0 \zeta_n.$$

Here

$$E_0 = \left(\frac{\hbar^2 F^2}{2m}\right)^{1/3} \,.$$

We have  $\zeta_1 \approx -2.337$ ,  $\zeta_2 \approx -4.088$ . The normalization constants  $A_n$  for each level are defined as

$$A_n^{-1} = \int_0^\infty dz \, |\chi_n(z)|^2 \, .$$

Normalized electron densities  $A_n |\chi_n(z)|^2$  are shown in Fig. 11.5. Each level creates a subband for the in-plane motion, the energy being

$$E_{n,\mathbf{k}} = E_n + E(\mathbf{k}) = E_n + \frac{\hbar^2 k^2}{2m}.$$

Note that the effective mass m is considerably smaller than the mass of a free electron.

### **Density of States**

The density of states  $g(\epsilon)$  is defined as number of states per the energy interval  $\epsilon$ ,  $\epsilon + d\epsilon$ . It is clear that

$$g(\epsilon) = \sum_{\alpha} \delta(\epsilon - \epsilon_{\alpha})$$

where  $\alpha$  is the set of quantum numbers characterizing the states. In the present case it includes the subband quantum number n, spin quantum number  $\sigma$ , valley quantum number

![](_page_6_Figure_1.jpeg)

Figure 11.5: Normalized electron densities  $A_n |\chi_n(z/\ell_F)|^2$  for the first (1) and second (2) subbands in a triangle potential with the slope F,  $\ell_F = (\hbar^2/2mF)^{1/3}$ .

v (for *n*-type materials), and in-plane quasimomentum **k**. If the spectrum is degenerate with respect to spin and valleys one can define the spin degeneracy  $\nu_s$  and valley degeneracy  $\nu_v$  to get

$$g(\epsilon) = \frac{\nu_s \nu_v}{(2\pi)^d} \sum_n \int d^d k \,\delta\left(\epsilon - E_{n,\mathbf{k}}\right) \,.$$

Here we calculate the number on states per unit volume, d being the dimension of the space. For 2D case we obtain easily

$$g(\epsilon) = \frac{\nu_s \nu_v m}{2\pi\hbar^2} \sum_n \Theta(\epsilon - E_n) \,.$$

Within a given subband it appears energy-independent. Since there can exist several subbands in the confining potential (see Fig. 11.6, inset), the total density of states can be represented as a set of steps, as shown in Fig. 11.6. At low temperature  $(kT \ll E_F)$  all the states are filled up to the Fermi level. Because of energy-independent density of states the sheet electron density is linear in the Fermi energy,

$$n_s = \mathcal{N} \frac{\nu_s \nu_v m E_F}{2\pi\hbar^2} + \text{const}$$

![](_page_7_Figure_1.jpeg)

Figure 11.6: Density of states for a quasi-2D system.

while the Fermi momentum in each subband can be determined as

$$k_{Fn} = \frac{1}{\hbar} \sqrt{2m(E_F - E_n)} \,.$$

Here  $\mathcal{N}$  is the number of transverse modes having the edges  $E_n$  below the Fermi energy. The situation is more complicated if the gas is confined into a narrow channel, say, along y-axis. In a similar way, the in-plane wave function can be decoupled as a product

$$\psi(\mathbf{r}) = \eta(y) \frac{1}{N} e^{ik_x x} \,,$$

where N is a proper normalization factor, the energy being

$$E_{n,s,k} = E_n + E_s(k_x) = E_n + E_s + \frac{\hbar^2 k_x^2}{2m}$$

Here  $E_{ns} \equiv E_n + E_s$  characterizes the energy level in the potential confined in both (z and y) directions. For square-box confinement the terms are

$$E_s = \frac{(s\pi\hbar)^2}{2mW^2} \,,$$

where W is the channel width, while for the parabolic confinement  $U(y) = (1/2)m\omega_0^2 y^2$  (typical for split-gate structures)

$$E_s = (s - 1/2)\hbar\omega_0.$$

It is conventional to introduce partial densities of states for the states with  $k_x > 0$  and  $k_x < 0$ ,  $g^{\pm}$ , respectively. We have,

$$g_s^+(\epsilon) = \frac{\nu_s \nu_v}{2\pi} \left(\frac{dE_s(k_x)}{dk_x}\right)^{-1} = \frac{\nu_s \nu_v \sqrt{m}}{2^{3/2} \pi \hbar} \frac{1}{\sqrt{\epsilon - E_{ns}}}.$$
 (11.2)

The total density of states is

$$g^{+}(\epsilon) = \frac{\nu_{s}\nu_{v}\sqrt{m}}{2^{3/2}\pi\hbar} \sum_{ns} \frac{\Theta(\epsilon - E_{ns})}{\sqrt{\epsilon - E_{ns}}}.$$
(11.3)

The energy dependence of the density of states for the case of parabolic confinement is shown in Fig. 11.7.

![](_page_8_Figure_1.jpeg)

Figure 11.7: Density of states for a quasi-1D system (solid line) and the number of states (dashed lines).

### Motion in a perpendicular magnetic field

2DEG in a perpendicular magnetic field gives an example of 0-dimensional electronic system. Indeed, according to the classical theory the Hamilton's function of a charged particle in an external electromagnetic field is

$$\mathcal{H} = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 + e\phi \,,$$

where  $\phi$  is the scalar and **A** is the vector potential of the field, and **p** is the generalized momentum of the particle. According to the rules of quantum mechanics, one should replace the canonical momentum **p** by the operator

$$\mathbf{p} 
ightarrow \hat{\mathbf{p}} = -i\hbar 
abla$$

and add also an extra spin term  $-\mu \mathbf{H}$  where  $\mu = \mu_B \hat{\mathbf{s}}/s$ . Here  $\mu_B = e/2mc$  is the *Bohr* magneton while  $\hat{\mathbf{s}}$  is the spin operator. Generally, interaction with periodic potential of the crystalline lattice leads to renormalization of the spin splitting  $\mu_B \to \mu^= g_f \mu_B$  where  $g_f$  is called the spectroscopic spin splitting factor.

Finally we get,

$$\mathcal{H} = \frac{1}{2m} \left( \hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right)^2 - \boldsymbol{\mu} \mathbf{H} + e\phi$$
  
=  $\frac{\mathbf{p}^2}{2m} - \frac{e}{2mc} (\mathbf{A} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{A}) + \frac{e^2 \mathbf{A}^2}{mc^2} - \frac{\mu}{s} \hat{\mathbf{s}} \cdot \mathbf{H} + e\phi.$ 

Since

$$\hat{\mathbf{p}} \cdot \mathbf{A} - \mathbf{A} \cdot \hat{\mathbf{p}} = -i\hbar \operatorname{div} \mathbf{A}$$

those operator commute if  $div \mathbf{A} = 0$ . It holds in a uniform field with

$$\mathbf{A} = \frac{1}{2}\mathbf{H} \times \mathbf{r} \,.$$

The wave function in a magnetic field is not uniquely defined: it is defined only within the gauge transform

$$\mathbf{A} \to \mathbf{A} + \nabla f$$
,  $\phi \to \phi - \frac{1}{c} \frac{\partial f}{\partial t}$ ,

where f is an arbitrary function of coordinates and time. Under such a transform only the phase of wave function is changed by the quantity  $ef/\hbar c$  that does not affect the observable quantities.

In classical mechanics, the generalized momentum of the particle is related to its velocity by the Hamilton equations,

$$m\mathbf{v} = \mathbf{p} - e\mathbf{A}/c$$
.

According to the quantum mechanics we arrive at a similar expression. However different components of velocity do not commute, the commutation rules being

$$\{ \hat{v}_x, \hat{v}_y \} = i(e\hbar/m^2 c) H_z , \{ \hat{v}_y, \hat{v}_z \} = i(e\hbar/m^2 c) H_x , \{ \hat{v}_z, \hat{v}_x \} = i(e\hbar/m^2 c) H_y .$$

That means that the particle cannot simultaneously have definite velocities in all three directions.

Let us determine the energy levels in a 3-dimensional system embedded into a uniform magnetic film with a vector potential

$$A_x = -Hy \,, \quad A_y = A_z = 0 \,.$$

The Hamiltonian then becomes

$$\mathcal{H} = \frac{1}{2m} \left( \hat{p}_x + \frac{eHy}{c} \right)^2 + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} - \frac{\mu}{s} \hat{s}_z H \,.$$

First, the operator  $\hat{s}_z$  commutes with the Hamiltonian. Thus z-component of spin is conserved and can be replaced by its eigenvalue  $\sigma$ . Thus one can analyze the Schrödinger equation for an ordinary coordinate function,

$$\frac{1}{2m}\left[\left(\hat{p}_x + \frac{eH}{c}y\right)^2 + \hat{p}_y^2 + \hat{p}_z^2\right]\psi - \frac{\mu}{s}\sigma H\psi = E\psi$$

It is naturally to search for solution in the form

$$\psi = e^{i(p_x x + p_z z)/\hbar} \phi(y) \,.$$

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The eigenvalues  $p_x$  and  $p_z$  take all values from  $-\infty$  to  $\infty$ . Since  $A_z = 0$  we get

$$p_z = m v_z$$

Thus the motion along magnetic field in 3D system is not quantized. For a motion in the xy-plane we have the following Schrödinger equation,

$$\phi'' + \frac{2m}{\hbar^2} \left[ \left( E + \frac{\mu\sigma}{s} H - \frac{p_z^2}{2m} \right) - \frac{1}{2} m \omega_c^2 (y - y_0)^2 \right] \phi = 0.$$
 (11.4)

Here

$$y_0 = -cp_x/eH = -a_H^2 k_x, \quad a_H = (c\hbar/eH)^{1/2}, \quad \omega_c = |e|H/mc.$$
 (11.5)

Since this equation is the same as the Schrödinger equation for a harmonic oscillator, we obtain

$$E = (n+1/2)\hbar\omega_c - (\mu\sigma/s)H + p_z^2/2m, \quad n = 0, 1, \dots$$
(11.6)

The first term gives discrete levels which corresponds to the finite motion in the xy-plane, they are called *Landau levels*. For an electron,  $\mu/s = -|e|\hbar/mc$ , and the energy spectrum reads as

$$E = \left(n + \frac{1}{2} + \sigma\right)\hbar\omega_c + \frac{p_z^2}{2m}.$$
(11.7)

The eigenfunctions  $\phi_n(y)$  are

$$\phi_n(y) = \frac{1}{\pi^{1/4} a_H^{1/2} \sqrt{2^n n!}} \exp\left[-\frac{(y-y_0)^2}{2a_H^2}\right] H_n\left[\frac{y-y_0}{a_H}\right].$$
 (11.8)

Here  $H_n$  is the Hermite polynomial.

In classical mechanics the motion in a magnetic field in xy-plane takes place in a circle about a fixed center. Here the conserved quantity  $y_0$  corresponds to y coordinate of the center of the circle. It is easy to see that the combination

$$x_0 = cp_y/eH + x$$

is also conserved, it commutes with the Hamiltonian. The quantity  $x_0$  corresponds to a classical x coordinate of the circle center. However, the operators  $\hat{y}_0$  and  $\hat{x}_0$  do not commute. That means that the coordinates  $x_0$  and  $y_0$  cannot take definite values simultaneously.<sup>1</sup>

One can ask: why the coordinates x and y are not equivalent? The reason is that the wave functions (11.8) correspond to the energy independent of  $k_y$ . Consequently, any function of the type

$$\sum_{k_x} C(k_x) \psi_{N,k_x,k_z}$$

$$y = y_0 + r_c(v_x/v_t), \quad x = x_0 - r_c(v_y/v_t).$$

<sup>&</sup>lt;sup>1</sup> In a classical mechanics, the radius of the circle is  $r_c = cmv_t/eH = v_t/\omega_c$ . Here  $v_t$  is the tangential component of the velocity. Thus we have,

corresponds to the same energy and one can chose convenient linear combinations to get correct asymptotic behavior.

To calculate the density of states in a magnetic field first we should count the number of the values  $k_y$  corresponding to the energy  $\varepsilon_{\alpha}$  (the so-called *degeneracy factor*). As usual, we apply cyclic boundary conditions along y and z -axes and get

$$k_x = \frac{2\pi}{L_x} n_y, \quad k_z = \frac{2\pi}{L_z} n_z.$$

At the same time, we assume that the solution exists only in the region

$$0 < y_0 < L_y$$

So, the degeneracy factor is

$$\frac{L_x}{2\pi} |k_x|^{\max} = \frac{L_x}{2\pi a_H^2} y_0^{\max} = \frac{L_y L_x}{2\pi a_H^2}.$$
(11.9)

This is very important relation which shows that one can imagine Landau states as cells with the area  $a_H^2$ . We will come back to this property later.

Now it is easy to calculate number of states in a 3D system treating the  $k_z$  variable as for the usual 1D motion

$$\frac{2|k_z|L_z}{2\pi} = \frac{2\sqrt{2m}L_z}{2\pi\hbar}\sqrt{\varepsilon - \hbar\omega_c(N+1/2)}$$

for each state with a given N. Finally, the total number of states per volume for a given spin is

$$Z_s(\varepsilon) = \sum_N Z_{sN}(\varepsilon) = \frac{2\sqrt{2m}}{(2\pi)^2 \hbar a_H^2} \sum_N \sqrt{\varepsilon - \hbar \omega_c (N+1/2)}$$

where one has to sum over all the values of N with non-negative  $\varepsilon - \hbar \omega_c (N+1/2)$ . The total number of sates is  $Z(\varepsilon) = 2Z_s(\varepsilon)$ . To get DOS one should to differentiate this equation with respect to  $\varepsilon$ . The result is

$$g_s(\varepsilon) = \frac{dZ(\varepsilon)}{d\varepsilon} = \frac{\sqrt{2m}}{(2\pi)^2 \hbar a_H^2} \sum_N \frac{\Theta[\varepsilon - \hbar \omega_c(N+1/2)]}{\sqrt{\varepsilon - \hbar \omega_c(N+1/2)}}.$$

Here

$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0; \\ 1/2 & \text{for } x = 0; \\ 0 & \text{for } x < 0 \end{cases}$$

is the Heaviside step function. To take the spin into account one should add the spin splitting  $\pm \mu_B g_f H$  to the energy levels. If we ignore the spin splitting we can assume spin degeneracy and multiply all the formulas by the factor 2. We take it into account automatically using  $g(\varepsilon) = 2g_s(\varepsilon)$ .

![](_page_12_Figure_1.jpeg)

Figure 11.8: Landau levels as functions of  $p_z$  (left panel) and of H (right panel). The Fermi level is assumed to be fixed by external conditions.

The behavior of the density of states could be interpreted qualitatively in the following way. The Landau levels as functions of magnetic field for a given value of  $p_z$  are shown in Fig. 11.8. As a function of magnetic field, they form the so-called *Landau fan*. The Fermi level is also shown. At low magnetic fields its dependence on magnetic field is very weak. We see that if magnetic field is small many levels are filled. Let us start with some value of magnetic field and follow the upper filled level N. As the field increases, the slopes of the "fan" also increase and at a given threshold value  $H_N$  for which

$$\varepsilon_N(H_N) = \epsilon_F$$

As the field increases the electrons are transferred from the N-th Landau level to the other ones. Then, for the field  $H_{N-1}$  determined from the equation  $\varepsilon_{N-1}(H_{N-1}) = \epsilon_F$  the (N-1)becomes empty. We get

$$H_N \approx \frac{m_c c \epsilon_F}{e \hbar} \frac{1}{N}, \quad \text{so} \quad \Delta\left(\frac{1}{H}\right) \approx \frac{e \hbar}{m_c c \epsilon_F}.$$

Here  $m_c$  is the so-called cyclotron effective mass which in the case of isotropic spectrum is the same as the density-of-states effective mass. We observe that DOS in a given magnetic field oscillated with the increase in energy just similar to the case of quasi 1D systems. Here Landau sub-bands play the same role as the modes of transverse quantization for quantum channels.

For a 2DEG the motion along z-direction is quantized, and instead of  $e^{ip_z z/\hbar}$  we have  $\chi_s(z)$ . The means that for each subband of spatial quantization we have a sharp Landau level, the density of states (per area) being

$$g(\epsilon) = \frac{\nu_v e H}{4\pi^2 \hbar^2 c} \sum_{n,s,\sigma} \delta(\epsilon - E_{n,s,\sigma}).$$

Thus the density of states has sharp maxima at the energy levels that is a feature of so-called 0-dimensional system. In real samples the peaks are smeared by disorder.

### 11.3 Degenerate and non-degenerate electron gas

At equilibrium the states are filled according to the Fermi function

$$f_0(\epsilon) = \frac{1}{\exp[(\epsilon - \mu)/kT] + 1},$$

where  $\mu$  is the chemical potential while k is the Boltzmann constant. The chemical potential is determined by the normalization to the total number of electrons as

$$n = \int_0^\infty g(\epsilon) f_0(\epsilon) \, d\epsilon$$

where n is the *electron density*. At zero temperature the chemical potential is called *the Fermi energy*,  $\epsilon_F$ . The graph of the Fermi function and its energy derivative is given in Fig. 11.9

![](_page_13_Figure_7.jpeg)

Figure 11.9: The Fermi distribution (solid line) and its energy derivative multiplied by kT (dashed line) for  $\zeta/kT = 10$ .

Since at T = 0

$$f_0(\epsilon) \approx \Theta(\epsilon - \zeta),$$

the Fermi energy is given by the equation

$$n = \int_0^{\epsilon_F} g(\epsilon) \, d\epsilon \,. \tag{11.10}$$

The limiting case T = 0 is actually means the inequality  $kT \ll \epsilon_F$  is met. In the opposite limiting case,  $kT \gg \epsilon_F$ , we get

$$f_0(\epsilon) \approx e^{(\zeta - \epsilon)/kT}, \quad n = e^{\zeta/kT} \int_0^\infty g(\epsilon) \, e^{-\epsilon/kT} \, d\epsilon$$

Thus,

$$f_0(\epsilon) = A(T) e^{-\epsilon/kT}, \quad \frac{1}{A(T)} = \frac{1}{n} \int_0^\infty g(\epsilon) e^{-\epsilon/kT} d\epsilon.$$
(11.11)

This distribution is called the *Boltzmann* one.

## 11.4 Relevant length scales

One can discriminate between several important length scales in low-dimensional systems. They are shown in the Table 11.1.

1 mm	
	Mean free path in the quantum Hall regime
100 <i>µ</i> m	1 1 0
100 µ	Mean free path/Phase relavation length
	In high-mobility semiconductor at $T < 4$ K
$10 \ \mu m$	
$1 \ \mu m$	
	Commercial semiconductor devices (1990)
100 nm	
	de Broglie wave length in semiconductors.
	Mean free path in polycrystalline metallic films
10 nm	~ ~ ~ ~
1 nm	
1 1111	de Proglie wave length in metals
	de brogne wave length in metals
	Distance between atoms
1 Å	

Table 11.1: A few relevant length scales. Note that 1  $\mu$ m = 10<sup>-6</sup> m = 10<sup>-4</sup> cm; 1 nm = 10<sup>-9</sup> m = 10 Å.

The above mentioned scales have the following physical meaning:

De Broglie wave length,  $\lambda$ . This length is defined as

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi}{k}$$

where p(k) is the typical electron momentum (wave vector). For Fermi gas the characteristic momentum is just the Fermi momentum. For the case of a single filled band in 2DEG,

$$\lambda = 2\pi/k_F = \sqrt{2\pi/n_s}$$

where  $n_s$  is the sheet density. For the Boltzmann gas,  $p \approx \sqrt{2mkT}$ , and

$$\lambda = \frac{2\pi\hbar}{\sqrt{2mkT}} \,.$$

Mean free path,  $\ell$ . This is a characteristic length between the collisions with impurities or phonons. It is defined as

$$\ell = v\tau_{\rm tr}$$

where v is the typical velocity while  $\tau_{tr}$  is the so-called *transport relaxation time*. It is defined as

$$\frac{1}{\tau_{\rm tr}} \propto \int d\theta \, \sin \theta \, W(\theta) \, (1 - \cos \theta)$$

where  $\theta$  is the scattering angle while  $W(\theta)$  is he scattering probability. Usually the transport is characterized by the *mobility* 

$$u = \frac{e\tau_{\rm tr}}{m}$$

The physical meaning of mobility is that a typical electron drift velocity acquired in an external electric field E is given by the relation

$$v_d = uE$$

**Phase-relaxation length**,  $L_{\varphi}$ . This is a specially quantum mechanical relaxation length which has no analogs in classical physics. Namely, classical motion can be described as evolution of the *probability* to find a particle at a given point at a given time. However, in quantum mechanics the state is characterized by the *wave function* which has a *phase*. The phase is important in the so-called interference phenomena, where the electron wave functions having different pre-history are collected at the same point. If the phases of the waves are not destroyed, a specific quantum interference phenomena can be observed and important. The phase-relaxation time,  $\tau_{\varphi}$ , describes relaxation of the phase memory.

It is clear that scattering against any static spin-independent potential cannot lead to the phase relaxation. Indeed, in any stationary potential the equations of motion are *time-reversible*. The only processes which can be responsible for phase relaxation are the ones which broke the symmetry with respect to time-reversal. Among them are inelastic scattering by phonons, electron-electron collisions, spin-flip processes, etc. An important feature of such processes is that an electron suffers many elastic collisions during a typical time  $\tau_{\varphi}$ . Since it moves *diffusively* a proper way to estimate the relevant length  $L_{\varphi}$  is as follows:

$$L_{\varphi} = \sqrt{D\tau_{\varphi}}$$

where  $D = (1/d)v\ell$  is the diffusion constant (d is the dimensionality of the electron gas).

**Thermal dephasing length**,  $L_T$ . The above mentioned relaxation process is relevant to the interference of the wave functions belonging to a single-electron state. However, interference can be also important for the interaction of *two electrons* having close

#### 11.4. RELEVANT LENGTH SCALES

energies. Indeed, if the energy difference between the electrons is  $\approx kT$  they travel almost coherently during the time  $\hbar/kT$ . Thus the characteristic length of coherent propagation is estimated as

$$L_T = \sqrt{\hbar D/kT}$$
.

Comparing mean free path  $\ell$  with characteristic dimensions of the system, L, one can discriminate between *diffusive*,  $\ell \ll L$  and *ballistic*,  $\ell \geq L$ , transport. Such a classification appears incomplete in the situation where different dimensions of the sample are substantially different. The situation is illustrated in Fig. 11.10 for the case where the length L of the sample is much larger than its width, W. If phase coherence is taken into account, the

![](_page_16_Figure_4.jpeg)

Figure 11.10: Electron trajectories for the diffusive  $(\ell < W, L)$ , quasi-ballistic  $(W < \ell < L)$ and ballistic  $(\ell > W, L)$  transport regimes. From [11].

scales  $L_{\varphi}$  and  $L_T$  become important, and the situation appears more rich and interesting. Mesoscopic conductors are usually fabricated by patterning a planar conductor that has one very small dimension to start with. Although some of the pioneering experiments in this field were performed using metallic conductors, most of the recent work has been based on the gallium arsenide (GaAs)-aluminum gallium arsenide (AlGaAs) material system. Some important parameters for such systems are shown in Fig. 11.11.

		GaAs(100)	Si (100)	Units
Effective Mass	m	0.067	0.19	$m_{\rm e} = 9.1 \times 10^{-28}  {\rm g}$
Spin Degeneracy	$g_{s}$	2	2	
Valley Degeneracy	g <sub>v</sub>	1	2	
Dielectric Constant	3	13.1	11.9	$\varepsilon_0 = 8.9 \times 10^{-12} \mathrm{F m^{-1}}$
Density of States Electronic Sheet	$\rho(E) = g_{\rm s}g_{\rm v}(m/2\pi\hbar^2)$	0.28	1.59	$10^{11} \mathrm{cm}^{-2} \mathrm{meV}^{-1}$
Density <sup>a</sup>	n	4	1-10	$10^{11} \mathrm{cm}^{-2}$
Fermi Wave Vector	$k_{\rm F} = (4\pi n_{\rm e}/q_{\rm e} q_{\rm v})^{1/2}$	1.58	0.56-1.77	$10^{6}  \mathrm{cm}^{-1}$
Fermi Velocity	$v_{\rm E} = \hbar k_{\rm E} / m$	2.7	0.34-1.1	$10^7  {\rm cm/s}$
Fermi Energy	$E_{\rm F} = (\hbar k_{\rm F})^2/2m$	14	0.63-6.3	meV
Electron Mobility <sup>a</sup>	$\mu_{e}$	$10^{4} - 10^{6}$	10 <b>4</b>	$cm^2/V \cdot s$
Scattering Time	$\tau = m\mu_e/e$	0.38-38	1.1	ps
Diffusion Constant	$D = v_{\rm F}^2 \tau/2$	140-14000	6.4–64	cm <sup>2</sup> /s
Resistivity	$\rho = (n_s e \mu_e)^{-1}$	1.6-0.016	6.3-0.63	kΩ
Fermi Wavelength	$\lambda_{\rm F} = 2\pi/k_{\rm F}$	40	112-35	nm
Mean Free Path	$l = v_{\rm F} \tau$	$10^2 - 10^4$	37-118	nm
Phase Coherence	-			
Length <sup>b</sup>	$l_{\phi} = (D\tau_{\phi})^{1/2}$	200	40-400	$nm(T/K)^{-1/2}$
Thermal Length	$l_{\rm T} = (\hbar D/k_{\rm B}T)^{1/2}$	330-3300	70-220	$nm(T/K)^{-1/2}$
Cyclotron Radius	$l_{\rm cycl} = \hbar k_{\rm F}/eB$	100	37-116	$nm(B/T)^{-1}$
Magnetic Length	$l_{\rm m}=(\hbar/eB)^{1/2}$	26	26	$nm(B/T)^{-1/2}$
	k <sub>F</sub> l	15.8-1580	2.1-21	
	$\omega_{\rm c} \tau$	1-100	1	( <i>B</i> /T)
	$E_{\rm F}/\hbar\omega_{\rm c}$	7.9	1-10	$(B/T)^{-1}$

Figure 11.11: Electronic properties of the 2DEG in GaAs-AlGaAs and Si inversion layers. From [10].

## Chapter 12

## **Ballistic transport**

### 12.1 Landauer formula

We start this chapter by a description of a very powerful method in physics of small systems - so-called *Landauer approach*.

The main principle of this approach is the assumption that the system in question is coupled to large reservoirs where all inelastic processes take place. Consequently, the transport through the systems can be formulated as a quantum mechanical *scattering problem*. Thus one can reduce the non-equilibrium transport problem to a quantum mechanical one.

Another important assumption is that the system is connected to reservoirs by *ideal* quantum wires which behave as waveguides for the electron waves. We start our analysis from the discussion of the properties of an ideal quantum wire.

### Ideal quantum wire

Consider 2 large reservoirs of electron gas reservoirs having the difference  $\delta n$  in the electron density and separated by a pure narrow channel. For small  $\delta n$  one can assume that there is a difference in a chemical potential,  $\delta \mu = \delta n/g(\epsilon_F)$ . In the following we shall use the Fermi level of non-biased system as the origin for the chemical potentials. So the difference between the chemical potential in  $\alpha$ -th reservoir will be denoted as  $\mu_{\alpha}$ .

If the channel is long and uniform, then the total current carried by the state characterized by a transverse mode n and a given direction of spin which propagates without scattering is

$$J_n = e \int \frac{dk_z}{2\pi\hbar} \frac{\partial \varepsilon_n(k_z)}{\partial k_z} = \frac{2}{2\pi\hbar} \int_{\epsilon_F + \mu_\alpha}^{\epsilon_F + \mu_\beta} d\varepsilon \frac{\partial \varepsilon_n(k_z)/\partial k_z}{|\partial \varepsilon_n(k_z)/\partial k_z|} = \frac{2}{\hbar} \delta \mu \,.$$

If we take into account electron spin and N transverse modes are open, then the conductance is given by the expression  $G = \frac{2e^2}{h}N$ .

We come to a very important conclusion: an *ideal* quantum wire has *finite* resistance  $h/2e^2N$  which is independent of the length of the wire.

As we have seen, even an ideal quantum wire has a finite resistance. That means a finite heat generation even in the absence of any inelastic processes inside the wire. Below we will discuss the physical picture of heat release by a current-carrying nanostructure (here we follow the considerations of Ref. [23]).

First of all let us specify what heat release is. It will be convenient to consider an isolated system. Therefore we will have in mind the following physical situation. There is a capacitor which is discharged through the conductor of interest. The product RC of the whole system, R and C being the resistance and capacitance respectively, is much bigger than any relaxation time characterizing the electron or phonon system of the conductor. This means that for all the practical purposes the conduction process can be looked upon as a stationary one. The total energy of the system,  $\mathcal{U}$ , is conserved, while its total entropy,  $\hat{\mathcal{S}}$ , is growing. The rate of heat generation is expressed through  $T\partial \hat{\mathcal{S}}/\partial t$ , where T is the temperature, i.e. through the applied voltage and characteristics of the nanostructure itself. This means that the result is independent of the assumption that the considered system is isolated, which is made only for the sake of derivation. This thermodynamically defined heat is generated in the classical reservoirs over the length having a physical meaning of the electron mean free path. That is the same mean free path that enters the Drude formula, which determines the conductivity of the reservoirs themselves, the amount of heat generated per second in both reservoirs being the same.

It is interesting to indicate that even purely elastic collisions can result in a heat generation although they of course cannot establish full equilibrium. This has a clear physical meaning. The amount of order in the electron distribution resulting in electric current can bring about mechanical work. For instance, one can let the current flow through a coil, and a magnetic rod can be drawn into the coil. In such a way the electrons transferring the current can execute a work on the rod. As a result of scattering, the amount of order in the electrons' distribution diminishes, and this means dissipation of mechanical energy into the heat. It has been shown that the heat release is symmetric in both reservoirs even if the scatterers in the system are asymmetric.

All the above considerations do not mean that the collisions that give the main contribution to the heat release, also establish *full equilibrium*. What equilibrium needs is inelastic collisions which transfer the energy of electrons taking part in charge transfer to other degrees of freedom, such as to other electrons and phonons. In particular, a *local equilibrium electron distribution* is established over the length scale determined by *electronelectron* interaction. Such a distribution can be characterized by a local electro-chemical potential and sometimes an electron temperature. The latter can in principle be measured by optical methods. On the other hand, the *equilibrium with respect to the lattice* is established at the scales of *electron-phonon* and *phonon-phonon* mean free paths. Only over those distances from the channel one can treat the results in terms of the true local temperature.

### Resistance of a quantum resistor

Consider a system shown in Fig. 12.1 consisting of a barrier connected to reservoirs by ideal quantum wires. If there is some reflection only a part of the current is transmitted.

![](_page_20_Figure_3.jpeg)

Figure 12.1: On the resistance of a quantum resistor.

In this case one can introduce the transmission probability of the mode  $n, T_n$ , to obtain (including spin degeneracy)

$$J = \frac{2}{h} \delta \mu \sum_{n=1}^{N} T_n$$

As a result,

$$G = \frac{2e^2}{h} \sum_{n=1}^{N} T_n = \frac{2e^2}{h} \operatorname{Tr} \mathbf{t} \mathbf{t}^{\dagger}.$$
 (12.1)

Here  $\mathbf{t}$  is the matrix of *scattering amplitudes* while the expression is called *two-terminal* Landauer formula.

This very important and looking simple formula was confusing during a long period. Indeed, this is the conductance which is measured between two reservoirs. Having in mind that the resistance of the connecting ideal wires (per one conducting mode) is  $h/2e^2$  we can ascribe to the scattering region the resistance

$$\frac{h}{2e^2} \left[ \frac{1}{T} - 1 \right] = \frac{h}{2e^2} \frac{R}{T},$$

where R is the reflection coefficient. Consequently, in the original formulation the quantum resistance was described as

$$G = \frac{2e^2}{h} \sum_{n=1}^{N} \frac{T_n}{1 - T_n} \,. \tag{12.2}$$

However, the quantity which is usually measured is given by Eq. (12.1).

Now we derive the Landauer formula for finite-temperature and so-called multichannel case when the leads have several transverse modes. Consider ideal wires which lead to a general elastic scattering system. Let each lead has the cross section A and have  $N_{\perp}$  transverse channels characterized by wave vectors  $k_i$  so that,

$$E_i + \frac{\hbar^2 k_i^2}{2m} = E_F$$

The incoming channels are fed from the electron baths with the same temperature and chemical potentials  $\mu_1, \mu_2, \ldots$ . The outgoing channels are fed up to thermal equilibrium population. We shall assume that the particles are absorbed in the outgoing baths. The sources are assumed to be *incoherent*, the differences  $\mu_1 - \mu_2$  are also assume small to yield linear transport. We introduce the scattering amplitudes  $t_{ij}$  for the transmission from *j*th incoming to *i*th outgoing channel. Reflection amplitudes  $r_{ij}$  are introduces in a similar way for reflection into the *i*th incoming channel. If we replace the incoming and outgoing channels, we denote the proper amplitudes by primes. In this way it is convenient to introduce  $2N_{\perp} \times 2N_{\perp}$  scattering matrix as

$$S = \left(\begin{array}{cc} r & t' \\ t & r' \end{array}\right) \,.$$

From the current conservation we must require unitarity while from time reversal symmetry  $S = \tilde{S}$ . Thus we have also  $SS^* = I$  where star stays for complex conjugation while tilde for transposition. In a magnetic field the Onsager relation requires  $S(H) = \tilde{S}(-H)$ .

It one defines the total transmission and reflection into *i*th channel as

$$T_i = \sum_j |t_{ij}|^2$$
,  $R_i = \sum_j |r_{ij}|^2$ .

then from unitarity condition we get

$$\sum_i T_i = \sum_i (1 - R_i) \,.$$

Since the densities of states in each channel are 1D like,  $g_i(E) = (\pi \hbar v_i)^{-1}$  we write the current through outgoing channels as

$$I = \frac{e}{\pi\hbar} \sum_{i} \int dE \left[ f_1(E) T_i(E) + f_2(E) R'(E) - f_2(E) \right]$$
$$= \frac{(\mu_1 - \mu_2)e}{\pi\hbar} \int dE \left( -\frac{\partial f}{\partial E} \right) \sum_{i} T_i(E) .$$

Thus the conductance becomes

$$G = \frac{2e^2}{h} \int dE \, \left(-\frac{\partial f}{\partial E}\right) \, \mathrm{Tr} \, \mathbf{t} \mathbf{t}^{\dagger} \, .$$

This is the *two-terminal* conductance measured between the *outside* reservoirs which includes contact resistances.

### Multiterminal resistance

For simplicity we shall discuss the case of zero temperature. Let us introduce the total transmission probability from the bath  $\alpha$  to the bath  $\beta$ ,

$$T_{\alpha \to \beta} = \sum_{n=1}^{N_{\alpha}} \sum_{m=1}^{N_{\beta}} |t_{\beta \alpha, mn}|^2 \,.$$

#### 12.1. LANDAUER FORMULA

Here  $N_i$  is the number of propagating modes in each lead connected to *i*th reservoir. Counting all the chemical potentials from the Fermi level, we see that the reservoir  $\alpha$  injects the current  $(2e/h)N_{\alpha}\mu_{\alpha}$  into the lead  $\alpha$ . The fraction  $T_{\alpha\to\beta}/N_{\alpha}$  is transmitted to the reservoir  $\beta$  while the fraction  $T_{\alpha\to\alpha}/N_{\alpha} = R_{\alpha}/N_{\alpha}$  is reflected back into reservoir  $\alpha$ . The the net current  $I_{\alpha}$  is given by the following set of equation,

$$\frac{h}{2e}I_{\alpha} + (N_{\alpha} - R_{\alpha})\mu_{\alpha} - \sum_{\beta \neq \alpha} T_{\beta \to \alpha}\mu_{\beta}.$$
(12.3)

Introducing vectors  $\vec{I}$  and  $\vec{\mu}$  with components  $I_{\alpha}$  and  $\mu_{\alpha}$ , respectively, we can write

$$\vec{I} = \hat{G}\vec{\mu}\,,\tag{12.4}$$

where the conductance matrix  $\hat{G}$  is defined as

$$G_{\alpha\beta} = \frac{2e^2}{h} \left[ \left( N_{\alpha} - R_{\alpha} \right) \delta_{\alpha\beta} - T_{\beta \to \alpha} \left( 1 - \delta_{\alpha\beta} \right) \right] \\ = \frac{2e^2}{h} \left[ N_{\alpha} \, \delta_{\alpha\beta} - T_{\beta \to \alpha} \right] \,.$$
(12.5)

Here we use the relation  $T_{\alpha \to \alpha} = R_{\alpha}$ . The sum of rows of this matrix is zero because of current conservation, the sum of the elements of each row also vanishes because if one changes all the chemical potentials by the same amount no current will be induced. Thus

$$N_{\alpha} - R_{\alpha} = \sum_{\beta \neq \alpha} T_{\beta \to \alpha} = \sum_{\beta \neq \alpha} T_{\alpha \to \beta} \,.$$

The equations (12.4) and (12.5) are called often the Landauer-Büttiker formalism. They

![](_page_22_Figure_10.jpeg)

Figure 12.2: On the resistance of 4-terminal device.

allow find, e. g. 4-terminal resistance. Indeed, we can put  $I_1 = -I_2 = I$ ,  $I_3 = I_4 = 0$ . Then  $\vec{I} = I\vec{j}$  where

$$\vec{j} = I \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$
.

Thus

$$\mathcal{R}_{34} = \frac{\mu_4 - \mu_3}{I} = \left(\hat{G}^{-1}\vec{j}\right)_4 - \left(\hat{G}^{-1}\vec{j}\right)_3.$$

Having in mind the properties of the scattering amplitudes we have,

$$T_{\alpha \to \beta}(H) = T_{\beta \to a}(-H)$$

that results in the reciprocity relation

$$\mathcal{R}_{\alpha\beta,\gamma\delta}(H) = \mathcal{R}_{\gamma\delta,\alpha\beta}(-H)$$
.

Here  $\mathcal{R}_{\alpha\beta,\gamma\delta}$  stands for the resistance measured for voltage contacts  $\gamma, \delta$  while the current passes through the contacts  $\alpha, \beta$ . Note that this relation works even in the case when the concept of local conductivity is not applicable. What we only need is linear response and absence of inelastic scattering inside the device under consideration.

One can easily generalize the above expressions for the case of finite temperatures by replacement of the element of  $\hat{G}$ -matrix by their thermal averages,

$$\langle A \rangle_T = \frac{\int_0^\infty dE \, g(E) \left(\partial f_0 / \partial E\right) A(E)}{\int_0^\infty dE \, g(E) \left(\partial f_0 / \partial E\right)}$$

## 12.2 Application of Landauer formula

### Point ballistic contact

The most clean system is the so-called quantum point contact (QPC) - short and narrow constrictions in 2d electron gas. A sketch of QPC is shown in Fig. 12.3 The conductance of

![](_page_23_Figure_11.jpeg)

Figure 12.3: A sketch of QPC formed by splitted gates.

QPC is quantized in the units of  $2e^2/h$ . The quantization is not that accurate as in quantum Hall effect (about 1%) because of non-unit transparencies  $T_n$  and finite temperature. It is interesting to compare quantum and classical behavior of QPC. In a classical picture one can write

$$J = W(\delta n) v_F \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{2\pi} \cos \alpha = \frac{1}{\pi} W v_F(\delta n) \,.$$

Thus the "diffusion constant" is

$$D_{eff} = \frac{J}{\delta n} = \frac{1}{\pi} W v_F \quad \rightarrow \quad G = e^2 g(\epsilon_F) D_{eff} = \frac{2e^2}{h} \frac{k_F W}{\pi}.$$

![](_page_24_Figure_1.jpeg)

Figure 12.4: Quantization of conductance of a point contact: Schematic picture (left) and experimental result (right).

![](_page_24_Figure_3.jpeg)

Figure 12.5: On the classical conductance of a point contact.

Note that the integer part of the quantity  $k_F W/\pi$  is just the number of occupied modes according to quantum mechanics.

### Series addition of quantum resistors

Assume that we have two obstacles in series. Let the wave with unit amplitude is incident to the region, the amplitude of the reflected wave is A while D is the amplitude the wave transmitted through device. The obstacles are connected by an ideal conductor, the phase shift of the wave along which being  $\phi$ . Let the wave emerging from the obstacle

![](_page_24_Figure_8.jpeg)

Figure 12.6: On series of quantum resistors.

1 is  $B \exp(kx - \omega t)$ . It reaches the obstacle 2 gaining the phase  $\phi$ , having the complex amplitude  $B \exp(i\phi)$ . The reverse wave C gains the phase  $-\phi$ . In this way we get the

following set of equations,

$$A = r_1 + t_1 C, \qquad B = t_1 + r'_1 C$$
$$Ce^{-i\phi} = r_2 Be^{i\phi}, \qquad D = t_2 Be^{i\phi}$$

Solving this equation we obtain,

$$D = \frac{e^{i\phi}t_1t_2}{1 - e^{2i\phi}r_2r_1'} \,,$$

that yields for the total transmittance:

$$T = |D|^2 = \frac{T_1 T_2}{1 + R_1 R_2 - 2\sqrt{R_1 R_2} \cos \theta}$$
(12.6)

where  $\theta = 2\phi + \arg(r_2r'_1)$ . The ratio between reflection and transmission which should be understood as a reduced resistance (in units  $h/2e^2$ ) of the system excluding wires is

$$\mathcal{G}^{-1} \equiv \frac{R}{T} = \left|\frac{A}{D}\right|^2 = \frac{R_1 + R_2 - 2\sqrt{R_1 R_2} \cos\theta}{T_1 T_2}, \quad \mathcal{G} \equiv \frac{hG}{2e^2}.$$
 (12.7)

This is a very strange formula. Assume that we made an ensemble of the systems which differ only by the distance between the obstacles, i. e. by the phase  $\phi$ . Let the distribution of  $\phi$  will be constant in the interval  $(0, 2\pi)$ . The averaging over  $\phi$  we get

$$\langle \mathcal{G}^{-1} \rangle = \frac{R_1 + R_2}{(1 - R_1)(1 - R_2)},$$

while the Ohm's law will provide

$$\mathcal{G}^{-1} = \frac{R_1}{1 - R_1} + \frac{R_2}{1 - R_2}.$$

As a result the Ohm's law survives only at small reflections.

Let us construct a chain of n resistors with very small reflections. Then the total reflection first increases linearly in n. Finally the total transmission becomes substantially less than 1. Now let us add a very good conductor to this chain. We get

$$\langle \mathcal{G}^{-1} \rangle_{n+1} = \frac{R_n + R}{T_n} = \langle \mathcal{G}^{-1} \rangle_n + \frac{R}{T_n}.$$

Thus an addition a good conductor increases the resistance by  $R/T_n > R$ . Such a behavior can be formulated as a "renormalization group"

$$\frac{1}{R}\frac{d}{dn}\langle \mathcal{G}^{-1}\rangle_n = \langle \mathcal{G}^{-1}\rangle_n + 1$$

Thus the average resistance grows exponentially with the length which has something to do with 1D localization. This considerations are not fully satisfactory because resistance is not the proper quantity to be averaged. Following Anderson, the proper quantity to be averaged is  $\ln(1 + \mathcal{G}^{-1})$ . Indeed,

$$1 + \mathcal{G}^{-1} = 1 + R/T = 1/T \quad \to \ln(1 + \mathcal{G}^{-1}) = -\ln T.$$

The quantity  $-\ln T$  plays the role of extinction exponent and it should be additive to successive scatterers is the relative phases are averaged out. We get this relation using

$$\int_0^{2\pi} d\theta \,\ln(a+b\cos\theta) = \pi \ln\frac{1}{2} \left[a + \sqrt{a^2 - b^2}\right]$$

So the exact scaling is given by the relation

$$\langle \ln\left(1+\mathcal{G}_n^{-1}\right)\rangle = n(Rh/2e^2).$$

### Parallel addition of quantum resistors.

Let us now discuss the parallel addition of two single-channeled quantum resistors. The geometry of the problem is shown in Fig. 12.7. All the phases and scattering effects along

![](_page_26_Figure_9.jpeg)

Figure 12.7: On the parallel addition of quantum resistances.

the branches are absorbed by the scattering parameters. Time-reversal symmetry requires  $t_i = t'_i$  while the current conservation requires

$$-t_i/t_i^{\prime\star} = r_i/r_i^{\prime\star}.$$

In the presence of Aharonov-Bohm flux  $\Phi$  through the loop, following from the gauge invariance the scattering amplitudes are renormalized as

$$t_1 \to t_1 e^{-i\theta}, t_1' \to t_1 e^{+i\theta}, t_2 \to t_2 e^{+i\theta}, t_2' \to t_2 e^{-i\theta}, r_i \to r_i, r_i' \to r_i'.$$

Here  $\theta = \pi \Phi / \Phi_0$ .

This point needs some more explanation. An Aharonov-Bohm flux  $\Phi$  through the opening can be represented as  $\oint \mathbf{A} \cdot d\mathbf{l}$  along the path circulating the opening. Here  $\mathbf{A}$  is the vector potential. One can eliminate this flux by a gauge transform

$$\psi' = \psi \exp\left[\frac{ie}{\hbar c} \sum_{j} \chi_j(\mathbf{r}_j)\right],$$

where  $\chi$  is defines as  $A_l = \nabla \chi$ . The transformed Schrödinger equation has  $A_l = 0$ . However, the price for this is that the transformed wave function,  $\psi'$ , does not satisfy periodic boundary conditions. When the electron coordinate is rotated once around the ring the phase of  $\chi'$  is changed by  $\delta \chi = 2\pi \Phi/\Phi_0$ . In our calculation this phase shift is absorbed into the expressions for the transition amplitudes.

To find the transmitted wave one has to determine 10 unknown amplitudes,  $x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, F$ . It can be done solving the set of matching equations at the scatterers and the triple connections. It is assumed that the connections do not introduce additional scattering and can be described by the unitary scattering matrix

$$S = \begin{pmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & -1/2 \\ -1/\sqrt{2} & -1/2 & 1/2 \end{pmatrix}.$$

Here  $S_{ii}$  denote the reflection amplitude of the *i*th channel while off-diagonal elements  $S_{ij}$  are the transition amplitudes from the channel *i* to *j*. The subscript 1 is used for left incoming channel and right outgoing channel. After rather long algebra made originally in Ref. [24] we arrive at the solution,

$$T \equiv |F|^2 = 4 \frac{\alpha + \beta \cos 2\theta}{\gamma + \delta \cos 2\theta + \epsilon \cos 4\theta}, \qquad (12.8)$$

where  $\alpha, \beta, \gamma, \delta, \epsilon$  are rather complicated functions of the scattering amplitudes,

$$\begin{aligned} \alpha &= |A|^2 + |B|^2, \ \beta = 2 \operatorname{Re}(AB^*), \ \gamma &= |D|^2 + |E|^2, \\ \delta &= 2 \operatorname{Re}(DC^* + EC^*), \ \epsilon = 2 \operatorname{Re}(DE^*), \\ A &= t_1^2 t_2 + t_2(r_1 - 1)(1 - r_1'), \ B &= t_2^2 t_1 + t_1(r_2 - 1)(1 - r_2'), \\ D &= E = t_1 t_2 \ C &= t_1^2 + t_2^2 - (2 - r_1 - r_2)(2 - r_1' - r_2'). \end{aligned}$$

This expression describes a rich physical picture. Even in the absence of magnetic field,  $\theta = 0$  the transmittance can be strongly dependent on the phases of the complex scattering amplitude. If we make one branch fully non-conducting,  $t_1 = 0$ , still the appropriate choice of phases of the reflection amplitudes  $r_1$  and  $r'_1$  can result either in T = 0 or T = 1. Indeed, in this case,

$$\begin{split} A &= t_2(r_1 - 1)(1 - r_1'), \ B = D = E = 0, \ C = t_2^2 - (2 - r_1 - r_2)(2 - r_1' - r_2'), \\ \alpha &= |t_2|^2 |(r_1 - 1)(1 - r_1')|^2, \ \beta = 0, \ \gamma = |t_2^2 - (2 - r_1 - r_2)(2 - r_1' - r_2')|^2, \\ \delta &= \epsilon = 0. \end{split}$$

As a result we get

$$T = \frac{|t_2|^2 |(r_1 - 1)(1 - r_1')|^2}{|t_2^2 - (2 - r_1 - r_2)(2 - r_1' - r_2')|^2}$$

Putting  $r_1 = 1$  we obtain T = 0. Thus we observe that non-conducting branch can influence the total conductance strongly.

Of course, all the discussed effects are due to interference. If the size of the system exceeds  $L_{\varphi}$  we come back to classical laws.

### 12.3 Additional aspects of ballistic transport

### Adiabatic point contacts

The results of first observations of conductance quantization were surprising. Indeed, from quantum mechanics it is well know that any sharp potential barrier produces oscillations in the transmission coefficient as a function of the energy. However, the experimental steps were more or less rectangular. An explanation of such a behavior was given in Ref. [25]. The authors showed that if the point contact has a smooth profile, i. e. if its width ddepends on the longitudinal coordinate x in a smooth way, then the T(E) dependence is very close to a perfect step. To make the results simple, let us consider a channel with rectangular confinement. Let us assume that we can separate the variables in an adiabatic way,  $\Psi_n(x, y) = \psi_n(x)\varphi_{n,x}(y)$ , and first solve the Schrödinger equation for a given width d. In this way we get the transverse wave functions

$$\varphi_{n,x}(y) = \sqrt{\frac{2}{d(x)}} \sin\left[\pi n \frac{2y + d(x)}{d(x)}\right]$$

The Schrödinger equation for the longitudinal motion has the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \epsilon_n(x)\psi = E\psi, \quad \epsilon_n(x) = \frac{\pi^2 n^2 \hbar^2}{2m[d(x)]^2}.$$

If the variation d(x) is smooth at the scale of de Broglie wave length,  $k_F^{-1}$ , the potential  $\epsilon_n(x)$  is semiclassical. Then one can use the semiclassical scheme for scattering problem and choose

$$\psi_n(x) = \sqrt{\frac{p_n(\infty)}{p_n(x)}} \exp\left[\frac{i}{\hbar} \int_0^x p_n(x') \, dx'\right], \quad p_n(x) = \sqrt{2m[E - \epsilon_n(x)]}.$$

The transmittance step depends occurs when the Fermi energy crosses the maximum of

![](_page_28_Figure_10.jpeg)

Figure 12.8: On the adiabatic quantum point contact.

the potential  $\epsilon_n(x)$  for the upper transverse mode, see Fig. 12.8. Expanding the potential near its maximum we get

$$\epsilon_n(x) = \epsilon_n(0) \left[ 1 - \left( \frac{\partial^2 d(x)}{\partial x^2} \right)_{x=0} \frac{x^2}{d} \right]$$

Since  $\partial^2 d/\partial x^2 = 2/\rho$  where  $\rho$  is the curvature radius of the center of constriction, we get the barrier as

$$U(x) = \epsilon_n(0) \left[ 1 - \frac{2x^2}{d\rho} \right]$$

The transmission through a parabolic barrier is known,

$$T(E) = \frac{1}{1 + \exp[-\pi^2 (kd_0/\pi - n_0)\sqrt{2\rho/d_0}]}.$$
(12.9)

Here  $d_0$  is the minimal width of the constriction,  $n_0$  is the number of upper level, while  $k = \hbar^{-1}\sqrt{2mE}$ . We observe that the shape of the step is almost n independent, the transition being sharp at  $\rho \ge d_0$ . It is important that the numerical factor  $\pi^2\sqrt{2}$  makes the transitions sharp even at  $R \sim d_0$ . The same numerical factor helps for the semiclassical condition to be valid. This criterion reads  $\pi^2\sqrt{2\rho/d_0} \gg 1$ . To make the motion through the contact ballistic the elastic mean free path should exceed  $\sqrt{\rho d_0}$ .

### **12.4** Electron-electron interaction in ballistic systems

### The case of pure 3D metal. Concept of Fermi liquid

Let us begin with the estimate of the electron-electron scattering in a Fermi gas. Suppose that we have a particle 1 outside the Fermi sea, see Fig. 12.9. If this particle interacts

![](_page_29_Figure_9.jpeg)

Figure 12.9: Scattering processes for electron-electron interaction.

with another one, 2, inside the Fermi sea both final states should be outside the Fermi sea (Pauli principle!). According to the momentum conservation law,

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2',$$

and, as we have seen,

$$p_1, p'_1, p'_2 > p_F; \qquad p_2 < p_F;$$

The momentum conservation law is shown graphically in the right panel of Fig. 12.9. One should keep in mind that the planes  $(\mathbf{p}_1, \mathbf{p}_2)$  and  $(\mathbf{p}'_1, \mathbf{p}'_2)$  are not the same, they are

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shown together for convenience. To get the escape probability for the particle 1 one should integrate over the intermediate momenta

$$W \propto \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2') (dp_2) (dp_1')$$

( $\mathbf{p}'_2$  is fixed by the momentum conservation). The energy conservation law actually determines the angle between  $\mathbf{p}'_1$  and  $\mathbf{p}'_2$  for given absolute values of these vectors. Consequently, the rest is to integrate over  $p_2 = |\mathbf{p}_2|$  and  $p'_1 = |\mathbf{p}_1|$ .

Let  $p_1$  be close to  $p_F$ . It means that all the momenta are close to  $p_F$  and the angles with the vector  $\mathbf{p}_1 + \mathbf{p}_2$  are almost the same. So let us assume cosines to be the same and from the relation between the projections write down

$$p_1' \approx p_1 + p_2 - p_2'$$

Now let us recall that  $p'_2 > p_F$ . Consequently,  $p'_1 < p_1 + p_2 - p_F$ . But at the same time,  $p'_1 > p_F$ . Thus

$$p_1 + p_2 - p_F > p_F$$
, or  $p_2 > 2p_F - p_1$ 

But from the Pauli principle, the upper limit for  $p_2$  is  $p_F$ . As a result, we come to the following chain of inequalities

$$0 > p_2 - p_F > p_F - p_1, \qquad 0 < p'_1 - p_F < (p_1 - p_F) + (p_2 - p_F).$$

Finally,

$$\int dp_2 \, dp'_1 = \int_{-\alpha_1}^0 d\alpha_2 \int_0^{\alpha_1 + \alpha_2} d\alpha'_1 = \frac{\alpha_1^2}{2}$$

where we have introduced  $\alpha_i = p_i - p_F$ . Now we should remember that  $\varepsilon - \epsilon_F = v_F(p - p_F)$ . So  $W \propto (\varepsilon - \epsilon_F)^2$ . The simplest way to estimate  $\tau$  is to use dimensionality approach. Indeed, the average potential and kinetic energies are of the order of  $\epsilon_F$ . Consequently, the only quantity which is proportional to  $(\varepsilon - \epsilon_F)^2$  and has the time dimensionality is

$$au \sim \frac{\hbar\epsilon_F}{(\varepsilon - \epsilon_F)^2}.$$

We came to important conclusion: if one is interested in quasiparticles near the Fermi level,  $|\varepsilon - \epsilon_F| \ll \epsilon_F$ , he can treat them as to near classical ones provided

$$\frac{\hbar}{(\varepsilon - \epsilon_F)\tau} \approx \frac{\varepsilon - \epsilon_F}{\epsilon_F} \ll 1.$$

The typical value for the quasiparticle energy is  $k_BT$ . This is why the electron-electron interaction can be treated in the leading approximation in a self-consistent approximation. The estimates above based on the conservation laws provide background of the theory of *Fermi liquid*. The essence of this concept is that the excitations in the vicinity of the the Fermi surface can be treated as *quasiparticles* which behave as particles with renormalized velocity. Consequently, the effects of electron-electron interaction are not crucially important in pure 3D systems.

### One-dimensional systems. Tomonaga-Luttinger liquid

For 1D interacting systems the above considerations are not valid because for a single branch linear dispersion near the Fermi points the energy spectrum is close to linear,  $E - E_F \approx v(p - p_F)$ . That means that the energy and momentum conservation laws are actually the same, and this is why they are not restrictive as in a 3D case. For this reason, the perturbative corrections describing even weak electron-electron interaction are divergent. A proper model for interactive 1D electrons is the so-called Tomonaga-Luttinger model. According to this model, collective electron modes (plasmons) with linear spectra are described by new, *boson* modes. Creation of a real electron in this model is equivalent to excitation an infinite number of plasmons. Because of that, the space and time dependence of density (and spin) correlation functions are substantially different from the ones for noninteracting systems. That manifests itself in various kinetic quantities. For example, the Drude conductivity is predicted to vary as power law with temperature.

The *Luttinger liquid* model which was previously used for 1D organic conductors now became important for high-mobility quantum wires, as well as for edge states under conditions of quantum Hall effect (see below). One can find a good review of this model in Ref. [16].

A 1D quantum wire is appropriately characterized by a conductance. It the absence of interactions, the conductance of an ideal single-mode quantum wire, adiabatically connected to leads, is quantized,  $G = 2e^2/h$ . In the presence of a scatterer, the conductance drops to  $G = 2e^2T/h$ , where T is the transmission coefficient.

The electron-electron interaction modifies dramatically the low-energy excitations in a quantum wire that leads to striking predictions for the transport. The new features manifest itself only if there is one (or several) scatterers inside the quantum wire - otherwise the correlation effects are canceled out at the contacts between the interacting quantum wire and non-interacting reservoirs. All that together leads to a rich and very interesting physical picture.

To get a flavor of the theory  $^1$  let us consider a spinless electrons hopping on 1D lattice with the Hamiltonian

$$\mathcal{H} = -t \sum_{j} c_{j}^{\dagger} c_{j+1} + \frac{V}{2} \sum_{j} c_{j}^{\dagger} c_{j} c_{j+1}^{\dagger} c_{j+1} + h.c. \qquad (12.10)$$

When the interaction V = 0 this Hamiltonian can be diagonalized as  $E_k = -t \cos k$ ,  $|k| < \pi$ . The low-energy excitation exist near  $\pm k_F$ . Consider a single particle excitation near  $+k_F$  where we remove one electron with  $k < k_F$  and place it into a free state with  $k + q > k_F$ . Then the energy of excitation is  $\hbar \omega_k = \hbar q v_F$ . Adding a similar state near  $-k_F$  we have a situation similar to phonons in one dimension. When the interaction is turned on this dispersion law remains, however the velocity is renormalized.

Linear spectrum implies a boson-like description. Mathematically in can be done using

<sup>&</sup>lt;sup>1</sup>Here we follow Ref. [16].

(Jordan-Wigner) canonical transform,

$$c_j = \exp\left(i\pi \sum_{k>j} c_k^{\dagger} c_k\right) b_j \,,$$

that keeps the Hamiltonian (12.10) in the same from with replacement  $c \to b$ . One can check that the *b* operators at different lattice points *commute*, and therefore they are *bosons*. Now, the boson operators can be approximately decoupled as  $b_j \to \sqrt{n_j} \exp(i\phi_j)$ ,  $n_j \equiv c_j^{\dagger}c_j$ Then we can proceed to continuum limit, focusing on scales long compared to the lattice constant. In this way we replace

$$\phi_j \to \phi(x) , \quad n_j \to \tilde{\rho}(x) ,$$

Extracting from the total electron density its average value,  $\rho_0 = k_F/\pi$  and introducing the "displacement" operator,  $\theta$ , as  $\tilde{\rho} - \rho_0 = \partial_x \theta(x)/\pi$  we arrive at the phonon-like commutation rule,

$$[\phi(x), \theta(x')] = \frac{i\pi}{2}\delta(x - x')$$

We observe that  $\partial_x \phi$  is the momentum conjugate to  $\theta$ . As a result, we arrive at the effective Hamiltonian,

$$\mathcal{H} = \frac{v}{2\pi} \left[ g(\partial_x \phi)^2 + g^{-1} (\partial_x \theta)^2 \right] \,. \tag{12.11}$$

From the commutation relations it can be seen that the Hamiltonian leads to the equation of motion

$$\partial_t^2 \theta = v^2 \partial_x^2 \theta$$

and a similar equation for  $\phi$ . the velocity v, as well as an additional dimensionless constant g depend on the strength of interaction. For non-interacting electrons,  $v = v_F$  and g = 1.

It is convenient to expand the original Fermion operator into two part corresponding to the motion around the points  $\pm k_F$ ,

$$\phi(x) \approx \psi_R + \psi_L = e^{k_F x} e^{i\Phi_R} + e^{-k_F x} e^{i\Phi_l 00}$$

where  $\Phi_{R/L} \equiv \phi \pm \theta$ . These two field commute with one another and satisfy the relations,

$$[\Phi_R(x), \Phi_R(x')] = -[\Phi_L(x), \Phi_L(x')] = i\pi \operatorname{sgn} (x - x').$$

The right and left moving electron densities can be reconstructed as

$$N_{B/L} = \pm \partial_x \Phi_{R/L}$$

Then we can rewrite the Hamiltonian (12.11) as

$$\mathcal{H} = \pi v_0 [N_R^2 + N_L^2 + 2\lambda N_R N_L] \tag{12.12}$$

with

$$v_0 = \frac{v}{2} \left( g + \frac{1}{g} \right) , \quad \lambda = \frac{1 - g^2}{1 + g^2} .$$

This Hamiltonian describes interacting system of right and left moving electrons. We identify  $v_0$  to  $v_F$ , the case g < 1 corresponds to  $\lambda > 0$  (repulsion) while g > 1 corresponds to attraction.

It is not straightforward how one can relate the parameters v and g with the original lattice model because we have only the effective theory for low-energy excitations. There are analytical expressions only for some models, e. g. for the case of long-range interaction comparing to the scale  $k_F^{-1}$ .

A very important point is that the parameter g has a physical meaning of dimensionless (in units  $e^2/h$  per spin) conductance of an infinite ideal Luttinger liquid. That can been traced introducing new variables,  $\phi_{R/L} = g\phi \pm \theta$ , that diagonalize the Hamiltonian (12.10) as

$$\mathcal{H} = \frac{\pi v}{g} \left( n_R^2 + n_L^2 \right) , \quad n_{R/L} = \pm \frac{1}{2\pi} \partial_x \phi_{R/L} . \tag{12.13}$$

The operators  $n_{R/L}$  correspond to the densities of right and left moving *interacting* electrons. The Hamiltonian allows chiral general solutions  $f(x \pm vt)$ . Now we can raise the chemical potential of right chiral mode  $n_R$  by an amount  $\mu_R$  Them  $\delta \mathcal{H} = -e\mu_R n_R$ , and minimizing (12.13) we get  $n_R = (ge/2\pi v)\mu_R$ . Since the additional current to the right is  $I_R = en_R v$  we get

$$G = g \frac{e^2}{h} \,. \tag{12.14}$$

As is was already mentioned, in a quantum wire it is impossible to couple only to one chiral mode. As a result, the d.c. conductance of a finite Luttinger liquid wire coupled to noninteracting leads appears the same as for noninteracting case. However, interaction effects can be revealed in a.c. experiments, as well as in the presence of some scatterers. Because of long-range correlations, the scatterers "undress" the excitations of interacting systems. As a result, may interesting and important effects can be observed. In particular, the interaction leads to a strong renormalization of density of states near an obstacle. For example, if the Luttinger liquid wire has a large barrier with low transmission coefficient  $T_0$  one can employ the results for the density of states in a semi-infinite liquid. That results in the nonlinear current-voltage curve at low temperatures.

$$I \propto T_0 |V|^{2(g-1)/g} V \to G(V) \equiv \frac{dI}{dV} \propto T_0 |V|^{2(g-1)/g}$$

Thus, got the repulsive case the linear conductance is *zero*. At finite temperature it does exist. However it is proportional to  $T^{2(g-1)/g}$ . It the case of weak scattering the results are substantially different.

There are several experiments where some deviations from the predictions of singleelectron theory were observed, such as unusual conductance quantization and anomalous temperature dependences. Unfortunately, the electron-electron correlations are effectively

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destroyed by disorder and electron-phonon scattering. Therefore, to observe the interaction effect one needs extremely pure samples and low temperatures. The results of such experiment is demonstrated in Fig. 12.10.

The concept of Luttinger liquid is specifically important for quantum Hall effect systems. We shall see that near the edges of a Hall bar a specific edge states appear which can be described by the above mentioned model. This system is much more pure that quantum wires, and interaction effects are crucially important. We are going to discuss quantum Hall systems later.

![](_page_35_Figure_1.jpeg)

(a) Wire preparation by cleaved edge overgrowth of GaAs-AlGaAs by MBE. The wire is fabricated by cleaving the specimen [see also panel (b)]. Edge states (d) form the quantum wire. Panels (e) and (f) show different charge distributions for different top voltages. The panel (c) shows a blowup of critical device region. The mean free path is estimated as 10  $\mu$ m, the length of the channel is about 2  $\mu$ m.

![](_page_35_Figure_3.jpeg)

Linear response conductance of a 2  $\mu$ m log wire in a 25 nm quantum well vs. the top-gate voltage ( $V_T$ ) measured at a temperature 0.3 K.Solid line is the measured conductance. The dashed curve is the measured conductance multiplied by an empirical factor 1.15. Inset: Linear response conductance of the last plateau for wires of different lengths fabricated consecutively along the edge of a single 25 nm cleaved edge overgrowth specimen. The numbers denote the wire lengths in microns.

Figure 12.10: Non-universal Conductance Quantization in Quantum wires [From A. Yacoby, *et al.*, Physical Review Letters, **77**, 4612 (1996).]

## Chapter 13

## **Tunneling and Coulomb blockage**

## 13.1 Tunneling

Modern technology allows to fabricate various structures involving tunneling barriers. One of the ways is a split-gate structure. Such a system can be considered as a specific example

![](_page_36_Figure_4.jpeg)

Figure 13.1: Split-gate structure allowing resonant tunneling.

of series connection of to obstacles. The complex amplitude of the wave transmitted through the whole system is

$$D = \frac{t_1 t_2 e^{i\phi}}{1 - e^{2i\phi} r_2 r_1'} = \frac{t_1 t_2 e^{i\phi}}{1 - e^{\theta} \sqrt{R_1 R_2}},$$
(13.1)

where  $\theta = 2\phi + \arg(r_2 r'_1)$ . It is clear that the transmittance

$$T = \frac{T_1 T_2}{1 + R_1 R_2 - 2\sqrt{R_1 R_2} \cos \theta}$$
(13.2)

is maximal at some specific value of  $\theta$  where  $\cos \theta = 1$ , the maximal value being

$$T_{\max} = \frac{T_1 T_2}{(1 - \sqrt{R_1 R_2})^2} \,. \tag{13.3}$$

This expression is specifically simple at  $T_1, T_2 \ll 1$ ,

$$T_{\max} = \frac{4T_1T_2}{(T_1 + T_2)^2}.$$
(13.4)

Thus we observe that two low-transparent barriers in series can have a unit transmittance if they have the same partial transparencies,  $T_1 = T_2 = T$ . The reason of this fact in quantum interference in the region between the barriers which makes wave functions near the barriers very large to overcome low transmittance of each barrier.

An important point is that the phase  $\theta$  gained in the system is a function of the electron energy. Thus near a particular value  $E^{(r)}$  defined by the equality

$$\cos\theta(E^{(r)}) = 0 \to \theta(E_k^{(r)}) = 2\pi k$$

one can expand  $\cos \theta$  as

$$1 - \frac{1}{2} \left(\frac{\partial \theta}{\partial E}\right)^2 \left(E - E^{(r)}\right)^2$$

Thus at low transmittance we arrive at a very simple formula of a Breit-Wigner type,

$$T \approx \frac{T_1 T_2}{(T_1 + T_2)^2 / 4 + (\theta')^2 (E - E^{(r)})^2} = \frac{\Gamma_1 \Gamma_2}{(\Gamma_1 + \Gamma_2)^2 / 4 + (E - E^{(r)})^2}.$$
(13.5)

Here we denote  $\theta' \equiv (\partial \theta / \partial E)_{E-E^{(r)}}$  and introduce  $\Gamma_i = T_i / |\theta'|$ .

The physical meaning of the quantities  $\Gamma_i$  is transparent. Let us assume that all the phase shift is due to ballistic motion of an electron between the barriers. Then,

$$\theta = 2ka = 2ah^{-1}\sqrt{2mE} \rightarrow \theta' = \frac{a}{\hbar}\sqrt{\frac{2m}{E}} = \frac{2a}{\hbar v}$$

where v is the electron velocity. As a result, the quantity  $\Gamma_i$  can be rewritten as  $\Gamma = \hbar \nu_a T_i$ , where  $\nu_a = v/2d$  is the frequency of oscillations inside the inter-barrier region, the so-called *attempt frequency*. Thus it is clear that  $\Gamma_i$  are the escape rates through *i*-th barrier.

To specify the transition amplitudes let us consider a 1D model for a particle in a well between two barriers. We are interested in the scattering problem shown in Fig. 13.2. To find a transmission probability one has to match the wave functions and their gradients at the interfaces 1-4 between the regions **A-C**. They have the following form

$e^{ikx} + re^{-ikx}$	in the region A;
$a_1 e^{\kappa_B x} + a_2 e^{-\kappa_B x}$	in the region B;
$b_1 e^{ikx} + b_2 e^{-ikx}$	in the region C;
$c_1 e^{\kappa_D x} + c_2 e^{-\kappa_D x}$	in the region D;
$te^{ikx}$	in the region E

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![](_page_38_Figure_1.jpeg)

Figure 13.2: On the resonant tunneling in a double-barrier structure.

Here

$$k = \hbar^{-1} \sqrt{2mE}$$
,  $\kappa_i = \sqrt{\kappa_{0i}^2 - k^2}$ ,  $\kappa_{0i} = \hbar^{-1} \sqrt{2mU_i}$ 

The transmission amplitude is given by the quantity t while the reflection amplitude – by the quantity r. In fact we have 8 equations for 8 unknowns  $(r, t, a_i, b_i, c_i)$ , so after some tedious algebra we can find everything. For a single barrier one would get

$$T(E) = \frac{4k^2\kappa^2}{\kappa_0^4\sinh^2(\kappa d) + 4k^2\kappa^2} \approx \frac{k^2\kappa^2}{\kappa_0^4}e^{-2\kappa d}.$$

Here d is the barrier's thickness. So the transparency exponentially decays with increase of the product  $\kappa d$ . The calculations for a double-barrier structure is tedious, so we consider a simplified model of the potential

$$U(x) = U_0 d[\delta(x) + \delta(x-a)]$$

In this case we have 3 regions,

$$e^{ikx} + re^{-ikx} \qquad x < 0.$$

$$A \sin kx + B \cos kx \qquad 0 < x < a,$$

$$te^{ik(x-a)} \qquad x > a$$
(13.7)

The matching conditions for the derivatives at the  $\delta$ -functional barrier has the form

$$\psi'(x_0+0) - \psi'(x_0-0) = \kappa^2 d\psi(x_0).$$
(13.8)

Here  $\kappa^2 = 2mU_0/\hbar^2$ . One can prove it by integration of the Schrödinger equation

$$(\hbar^2/2m)\nabla^2\psi + U_0d\delta(x-x_0)\psi = E\psi$$

around the point  $x_0$ . Thus we get the following matching conditions

$$B = 1 + r,$$
  

$$kA - ik(1 - r) = \kappa^2 a(1 + r),$$
  

$$A \sin ka + B \cos ka = t,$$
  

$$ikt - k(A \cos ka - B \sin ka) = t\kappa^2 a.$$

First one can easily see that there is a solution with zero reflectance, r = 0. Substituting r = 0 we get the following requirement for the set of equation to be consistent

$$k = k_0, \quad \tan k_0 a = -\frac{2k_0}{\kappa^2 d}.$$
 (13.9)

We immediately observe that at  $k = k_0$  the total transmission amplitude |t| = 1 and there is no reflection.

Equation (13.9) can be rewritten as

$$\tan z = -\eta z$$
, where  $\eta \equiv 2/\kappa^2 da = \hbar^2/mU_0 ad$ .

The solution of this equation can be found graphically. For "strong barriers",  $\eta \ll 1$  the solution tends to

$$k_0 a = s\pi$$
,  $s = 0, \pm 1, ...$ 

while for "weak" barriers

$$k_0 a = (s+1/2)\pi$$
.

Physically, that means that an electron gains the phase  $2\pi s$  during its round trip (cf. with optical interferometer). Thus two barriers in series can have perfect transparency even if the transparency of a single barrier is exponentially small. The physical reason is *quantum interference*.

The condition (13.9) defines the energy

$$E_0 = \frac{\hbar^2 k_0^2}{2m}$$

where the transparency is maximal. Near the peak one can expand all the quantities in powers of

$$k - k_0 \approx \frac{E - E_0}{(\partial E / \partial k)_{k_0}} \approx k_0 \frac{E - E_0}{2E_0}$$

The result for a general case can be expressed in the Breit-Wigner form

$$T(E) = \frac{\Gamma_L \Gamma_R}{(E - E_0)^2 + \frac{1}{4} (\Gamma_L + \Gamma_R)^2}.$$

Here  $\Gamma_{L(R)}/\hbar$  are the *escape rates* for the electron inside the well to the left(right) lead. They are given by the attempt frequency  $v_0/2a = \hbar k_0/2ma$  times the transparency of a given barrier.

Of course, if voltage across the system is zero the total number of electrons passing along opposite directions is the same, and the current is absent. However, in a biased system we obtain the situation shown in Fig. 13.3. Negative differential conductance,  $dJ/dV \leq 0$ , allows one to make a generator. One can also control the system moving the level  $E_0$  with respect to the Fermi level by the gate voltage. In this way, one can make a *transistor*.

Commercial resonant tunneling transistors are more complicated than the principle scheme described above. A schematic diagram of a real device is shown in Fig. 13.4. In

![](_page_40_Figure_1.jpeg)

![](_page_40_Figure_2.jpeg)

this device resonant tunneling takes place through localized states in the barrier. There exist also transistors with two quantum wells where electrons pass through the resonant levels in two quantum wells from the emitter to collector if the levels are aligned. The condition of alignment is controlled by the collector-base voltage, while the number of electrons from emitter is controlled by the base-emitter voltage.

## 13.2 Coulomb blockade

Now let us discuss a specific role of Coulomb interaction in a mesoscopic system. Consider a system with a dot created by a split-gate system (see above).

If one transfers the charge Q from the source to the grain the change in the energy of the system is

$$\Delta E = QV_G + \frac{Q^2}{2C}.$$

Here the first item is the work by the source of the gate voltage while the second one is the energy of Coulomb repulsion at the grain. We describe it by the effective capacitance C to take into account polarization of the electrodes. The graph of this function is the parabola with the minimum at

$$Q = Q_0 = -CV_G,$$

So it can be tuned by the gate voltage  $V_G$ . Now let us remember that the charge is transferred by the electrons with the charge -e. Then, the energy as a function of the number n of electrons at the grain is

$$\Delta E(n) = -neV_G + \frac{n^2e^2}{2C}.$$

Now let us estimate the difference

$$\Delta E(n+1) - \Delta E(n) = -eV_G + n\frac{e^2}{C}.$$

![](_page_41_Figure_1.jpeg)

Figure 13.4: Schematic diagram of a Si MOSFET with a split gate (a), which creates a potential barrier in the inversion layer (b). In the right panel oscillations in the conductance as a function of gate voltage at 0.5 K are shown. They are attributed to resonant tunneling through localized states in the barrier. A second trace is shown for a magnetic field of 6 T. From T. E. Kopley *et al.*, Phys. Rev. Lett. **61**, 1654 (1988).

We observe that at certain values of  $V_G$ ,

$$V_{Gn} = n \frac{e}{C} \,, \tag{13.10}$$

the difference vanishes. It means that only at that values of the gate voltage resonant transfer is possible. Otherwise one has to pay for the transfer that means that only inelastic processes can contribute. As a result, at

$$k_B T \le \frac{e^2}{2C}$$

the linear conductance is exponentially small if the condition (13.10) is met. This phenomenon is called the Coulomb blockade of conductance.

As a result of the Coulomb blockade, electron tunnel *one-by-one*, and the conductance vs. gate voltage dependence is a set of sharp peaks. That fact allows one to create a so-called *single-electron transistor* (SET) which is now the *most sensitive* electrometer. Such a device (as was recently demonstrated) can work at room temperature provided the capacitance (size!) is small enough.

Coulomb blockade as a physical phenomenon has been predicted by Kulik and Shekhter [26]. There are very good reviews [13, 14, 15] about single-change effects which cover both principal and applied aspects. Below we shall review the simplest variant of the theory, so called "orthodox model".

### A simple theory of single charge tunneling

For simplicity, let us ignore discrete character of energy spectrum of the grain and assume that its state is fully characterized by the number n of excess electrons with respect to

an electrically neutral situation. To calculate the energy of the systems let us employ the equivalent circuit shown in Fig. 13.5. The left (emitter) and right (collector) tunnel

![](_page_42_Figure_2.jpeg)

junctions are modeled by partial resistances and capacitances.

The charge conservation requires that

$$-ne = Q_e + Q_c + Q_g = C_e(V_e - U) + C_c(V_c - U) + C_g(V_g - U),$$
 (13.11)

where U is the potential of the grain. The effective charge of the grain is hence

$$Q = CU = ne + \sum_{i=e,c,g} C_i V_i , \quad C \equiv \sum_i C_i .$$

This charge consists of 4 contributions, the charge of excess electrons and the charges induced by the electrodes. Thus, the electrostatic energy of the grain is

$$E_n = \frac{Q^2}{2C} = \frac{(ne)^2}{2C} + \frac{ne}{C} \sum_i C_i V_i + \frac{1}{2C} \left(\sum_i C_i V_i\right)^2.$$
 (13.12)

The last item is not important because it is *n*-independent. In the stationary case, the currents through both junctions are the same. Here we shall concentrate on this case. In the non-stationary situation, an electric charge can be accumulated at the grain, and the currents are different.

To organize a transport of one electron one has to transfer it first from emitter to grain and then from grain to collector. The energy cost for the first transition,

$$E_{n+1} - E_n = \frac{(2n+1)e^2}{2C} + \frac{e}{C}\sum_i C_i V_i$$
(13.13)

must be less than the voltage drop  $eV_e$ . In this way we come to the criterion

$$E_n - E_{n+1} + eV_e \ge 0. (13.14)$$

![](_page_42_Figure_16.jpeg)

In a similar way, to organize the transport from grain to collector we need

$$E_{n+1} - E_n - eV_c \ge 0. (13.15)$$

The inequalities (13.15) and (13.14) provide the relations between  $V_e, V_c$  and  $V_g$  to make the current possible. For simplicity let us consider a symmetric system, where

$$G_e = G_c = G, \ C_e = C_c \approx C/2 \ (C_g \ll C), \ V_e = -V_c = V_b/2$$

where  $V_b$  is bias voltage. Then we get the criterion,

$$V_b \ge (2n+1)|e|/C - 2(C_g/C)V_g$$
.

We observe that there is a threshold voltage which is necessary to exceed to organize transport. This is a manifestation of *Coulomb blockade*. It is important that the threshold linearly depends on the gate voltage which makes it possible to create a transistor. Of course, the above considerations are applicable at zero temperature.

The current through the emitter-grain transition we get

$$I = e \sum_{n} p_n \left[ \Gamma_{e \to g} - \Gamma_{g \to e} \right] \,. \tag{13.16}$$

Here  $p_n$  is the stationary probability to find n excess electrons at the grain. It can be determined from the balance equation,

$$p_{n-1}\Gamma_{n-1}^{n} + p_{n+1}\Gamma_{n+1}^{n} - \left(\Gamma_{n}^{n-1} + \Gamma_{n+1}^{n}\right)p_{n} = 0.$$
(13.17)

Here

$$\Gamma_{n-1}^{n} = \Gamma_{e \to g}(n-1) + \Gamma_{c \to g}(n-1);$$
(13.18)

$$\Gamma_{n+1}^{n} = \Gamma_{g \to e}(n+1) + \Gamma_{g \to c}(n+1).$$
(13.19)

The proper tunneling rates can be calculated from the golden rule expressions using tunneling transmittance as perturbations. To do that, let us write down the Hamiltonian as

$$\begin{aligned} \mathcal{H}_{0} &= \mathcal{H}_{e} + \mathcal{H}_{g} + \mathcal{H}_{ch} + \mathcal{H}_{bath} \,; \\ \mathcal{H}_{e,c} &= \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} \,, \\ \mathcal{H}_{g} &= \sum_{\mathbf{q}\sigma} \epsilon_{\mathbf{q}} c_{\mathbf{q}\sigma}^{\dagger} c_{\mathbf{q}\sigma} \,, \\ \mathcal{H}_{ch} &= (\hat{n} - Q_{0})/2C \,, \quad \hat{n} = \sum_{\mathbf{q}\sigma} c_{\mathbf{q}\sigma}^{\dagger} c_{\mathbf{q}\sigma} - N^{+} \end{aligned}$$

Here  $\mathcal{H}_{\text{bath}}$  is the Hamiltonian for the thermal bath. We assume that emitter and collector electrodes can have different chemical potentials.  $N^+$  is the number of positively charged

ions in the grain. To describe tunneling we introduce the tunneling Hamiltonian between, say, emitter and grain as

$$\mathcal{H}_{e\leftrightarrow g} = \sum_{\mathbf{k},\mathbf{q},\sigma} T_{\mathbf{k}\mathbf{q}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{q}\sigma} + \text{h.c.}$$

Applying the golden rule we obtain

$$\Gamma_{e \to g}(n) = \frac{G_e}{e^2} \int_{-\infty}^{\infty} d\epsilon_k \int_{-\infty}^{\infty} d\epsilon_q f_e(\epsilon_k) [1 - f_g(\epsilon_q)] \,\delta(E_{n+1} - E_n - eV_e) \,.$$

Here we have introduced the tunneling conductance of e - g junction as

$$G_e = (4\pi e^2/\hbar) g_e(\epsilon_F) g_g(\epsilon_F) \mathcal{V}_e \mathcal{V}_g \langle |T_{\mathbf{kq}}|^2 \rangle \,.$$

along the Landauer formula,  $\mathcal{V}_{e,g}$  being the volumes of the lead and grain, respectively. In this way one arrives at the expressions

$$\Gamma_{e \to g}(n, V_e) = \Gamma_{g \to e}(-n, -V_e) = \frac{2G_e}{e^2} \mathcal{F}(\Delta_{+, e}); \qquad (13.20)$$

$$\Gamma_{g \to c}(n, V_c) = \Gamma_{c \to g}(-n, -V_c) = \frac{2G_c}{e^2} \mathcal{F}(\Delta_{-,c}).$$
(13.21)

Here

$$\mathcal{F}(\epsilon) = \frac{\epsilon}{1 + \exp(-\epsilon/kT)} \to \epsilon \,\Theta(\epsilon) \text{ at } T \to 0,$$

while

$$\Delta_{\pm,\mu}(n) = E_n - E_{n\pm 1} \pm eV_{\mu} = \frac{1}{C} \left[ \frac{e^2}{2} \mp en \mp e \sum_i C_i V_i \right] \pm eV_{\mu}$$

is the energy cost of transition. The temperature-dependent factor arise from the Fermi occupation factor for the initial and final states, physically they describe thermal activation over Coulomb barrier. The results of calculation of current-voltage curves for a symmetric transistor structure are shown in Fig. 13.6. At low temperatures and low bias voltages, VC/e < 1, only two charge states play a role. At larger bias voltage, more charge states are involved. To illustrate this fact, a similar plot is made for symmetrically biased transistor,  $V_e = -V_g = V/2$ , for different values of  $Q_0$ , Fig. 13.7.

#### Cotunneling processes

As we have seen, at low temperature the sequential tunneling can be exponentially suppressed by Coulomb blockade. In this case, a higher-order tunneling process transferring electron charge coherently through two junctions can take place. For such processes the excess electron charge at the grain exists only virtually.

A standard next-order perturbation theory yields the rate

$$\Gamma_{i \to f} = \frac{2\pi}{\hbar} \left| \sum_{\psi} \frac{\langle i | \mathcal{H}_{\text{int}} | \psi \rangle \langle \psi | \mathcal{H}_{\text{int}} | i \rangle}{E_{\psi} - E_{i}} \right|^{2} \delta(E_{i} - E_{f}).$$

Two features are important.

![](_page_45_Figure_1.jpeg)

Figure 13.6: The current of a symmetric transistor as a function of gate and bias voltage at T = 0 (from the book [5]).

- There are 2 channel which add coherently: (i)  $e \to g, g \to c$  with the energy cost  $\Delta_{-,e}(n+1)$ , and (ii)  $g \to c, e \to g$  with the energy cost  $\Delta_{+,c}(n-1)$ .
- The leads have macroscopic number of electrons. Therefore, with overwhelming probability the outgoing electron will come from a different state than the one which the incoming electron occupies. Hence, after the process an *electron-hole excitation* is left in the grain.

Transitions involving different excitations are added incoherently, the result being

$$\Gamma_{\rm cot} = \frac{\hbar G_e G_c}{2\pi e^4} \int_e d\epsilon_k \int_g d\epsilon_q \int_g d\epsilon_{q'} \int_c d\epsilon_{k'} f(\epsilon_k) [1 - f(\epsilon_q)] f(\epsilon_{q'}) [1 - f(\epsilon_{k'})] \\ \times \left[ \frac{1}{\Delta_{-,e}(n+1)} + \frac{1}{\Delta_{+,c}(n-1)} \right]^2 \delta(eV + \epsilon_k - \epsilon_q + \epsilon_{q'} - \epsilon_{k'}).$$

At T = 0 the integrals can be done explicitly, and one obtains

$$\Gamma_{\rm cot} = \frac{\hbar G_e G_c}{12\pi e} \left[ \frac{1}{\Delta_{-,e}(n+1)} + \frac{1}{\Delta_{+,c}(n-1)} \right]^2 V^3 \quad \text{for} \quad eV \ll \Delta_i \,.$$

As a result, the current appears proportional to  $V^3$  that was observed experimentally. The situation is not that simple for the degenerate case when  $\Delta_i = 0$ . In that case the integrals are divergent and the divergence must be removed by a finite life time of a state. A detailed treatment of that case is presented in the book [5].

There is also a process when an electron tunnels through the system leaving no excitations in the grain. The probability of such *elastic cotunneling* has a small factor  $(g_g \mathcal{V}_g)^{-1}$ . However, it leads to the current, proportional to V, thus it can be important at very low bias voltage.

![](_page_46_Figure_1.jpeg)

Figure 13.7: The current of asymmetric transistor,  $G_e = 10G_c$ , as a function of bias voltage at T = 0 and different  $Q_0.e = 0$ , 0.25 and 1 (from the book [5]). At  $Q_0 = 0$  the Coulomb blockade is pronounced, while at  $Q_0/e = 0.5$  the current-voltage curve is linear at small bias voltage. The curves of such type are called the *Coulomb staircase*.

#### **Concluding remarks**

There are many experiments where Coulomb-blockaded devices are investigated. Probably most interesting are the devices where tunneling takes place through a small quantum dot with discrete spectrum. An example of such device is shown in Fig. 13.8. The linear conductance of such a structure as a function of the gate electrode C is shown in Fig. 13.9 An important point is that at present time the devices can be fabricated so small that the criterion  $kT \leq e^2/C$  can be satisfied at room temperatures. Now several room temperature operating Coulomb blockade devices were reported. Among them are devices consisting of large molecules with the probes attached in different ways. This is probably a starting point for new extremely important field - molecular electronics. Such devices are extremely promising both because they are able to operate at room temperatures and because they will allow high integration. This is one of important trends. Another one concerns with single-electron devices which include superconducting parts. There is a lot of interesting physics regarding transport in such systems.

![](_page_47_Figure_1.jpeg)

Figure 13.8: (a) A typical structure of a quantum dot. The depleted (shaded) areas are controlled by electrodes 1-4, C, and F. Electrode C also controls the electrostatic potential in the dot. (b) a model of a quantum dot. From [7].

![](_page_47_Figure_3.jpeg)

Figure 13.9: Conductance of a quantum dot vs. the voltage of gate electrode C. From L. P. Kouwenhoven *et al.*, Z. Phys. B **85**, 367 (1991).

## Chapter 14

## Quantum Hall Effect and Adiabatic Transport

### 14.1 Ordinary Hall effect

In a magnetic field electrons' trajectories are curved because of Lorentz force. As a result,

$$\mathbf{j} = \sigma_0 \left( \mathbf{E} + [\mathbf{j} \times \mathbf{H}] / nec \right), \quad \sigma_0 = n e^2 \tau / m.$$

One can solve this vector equation to obtain the resistivity tensor

$$\hat{\rho} = \begin{bmatrix} \rho_0 & H/enc \\ -H/enc & \rho_0 \end{bmatrix}, \quad \rho_0 = 1/\sigma_0$$

The inversion of this tensor gives the *conductivity tensor* with the components (in 2d case)

$$\sigma_{xx} = \frac{\sigma_0}{1 + (\omega_c \tau)^2}, \quad \sigma_{xy} = \frac{nec}{H} + \frac{1}{\omega_c \tau} \sigma_{xx}.$$
(14.1)

There is a striking similarity between the quantization of the conductance of a ballistic channel in units of  $e^2/h$  and of the Hall conductance.

## 14.2 Integer Quantum Hall effect - General Picture

In the quantum case one faces the Landau levels. We have seen that the number of states per unit area for a filled Landau level is

$$n_H = 1/2\pi a_H^2 = eH/ch$$
.

Usually the *filling factor* 

$$\nu = n/n_H$$

for a fractionally filled level is introduced. The total degeneracy of a Landau level in the absence of disorder is  $N = An_B = A/2\pi a_H^2$  where A is the sample area. The average density of states is  $An_B/\hbar\omega_c$ .

If one expresses the Hall component of the conductivity tensor through the filling factor  $\nu$  and assumes  $\omega_c \tau \to \infty$  he obtains  $\sigma_{xy} = \nu e^2/h$ . This result seems very fundamental. Indeed, according to the electrodynamics, one can eliminate the electric field on a *spatially-homogeneous* system by assuming a drift velocity

$$\mathbf{v} = c \left[ \mathbf{E} \times \mathbf{H} \right] / H^2 \,.$$

Thus, the result seems fundamental independently whether classical of quantum mechanics is discussed.

Experimentally, the following picture was observed. It is clear that only violation of

![](_page_49_Figure_6.jpeg)

Figure 14.1: Schematic dependence of Hall resistance on filing factor.

the translational invariance can lead to such a picture. Thus one has to consider either impurities, or edges.

The generally accepted picture is as follows. Impurities violate the translational invariance. As a result,  $p_y$  is not a good quantum number any more, and Landau levels smear into sub-bands. The crucial point that the most of the states are *localized* and cannot carry the current.

To make the analysis as simple as possible let us discuss a 2d electrons in crossed electric and magnetic fields ( $\mathbf{E} \parallel \mathbf{x}, \mathbf{H} \parallel \mathbf{z}$ ) and a single impurity at the point  $\mathbf{r}_0 = \{x_0, y_0\}$ . As we have seen (see Sec. 6.9, *Shubnikov-de Haas effect*), the weak electric field leads to the energy shift  $p_y v$  where v = cE/H is the drift velocity in y-direction, as well as to the shift in the center-of-motion co-ordinate is shifted in x-direction by  $v/\omega_c$ . Using the corresponding states as a basis, we can now expand the exact wave function as

$$\Psi = \sum_{np_y} c_{np_y} \psi_{np_y}(\mathbf{r}) \,.$$

We get

$$\sum_{np_y} c_{np_y} \left[ \hat{\mathcal{H}}_0 + V \right] \psi_{np_y} = \sum_{np_y} c_{np_y} \left[ E_{np_y} + V \right] \psi_{np_y} = E \sum_{np_y} c_{np_y} \psi_{np_y}$$

Let us now express the potential as

$$V = \lambda \delta(\mathbf{r} - \mathbf{r}_0)$$

where  $\lambda$  is a proper coupling constant. Then one can write

$$c_{npy} = \lambda \frac{\psi_{npy}^*(\mathbf{r}_0)\Psi(\mathbf{r}_0)}{E - E_{npy}}$$

Now we recall that  $\sum_{np_y} c_{np_y} \psi_{np_y}(\mathbf{r}_0) = \Psi(\mathbf{r}_0)$ . Substituting the previous expression into this equation and dividing by  $\Psi(\mathbf{r}_0)$ , we get the exact condition for eigen energy

$$\frac{1}{\lambda} = \sum_{n, p_y} \frac{|\psi_{n, p_y}(\mathbf{r}_0)|^2}{E - E_{n, p_y}} \,.$$

The right hand side of this equation as a function of the energy is shown below One can

![](_page_50_Figure_10.jpeg)

Figure 14.2: Formation of localized states in 2DEG in magnetic field.

find from this equation one completely localized state for each Landau level, its energy shift being proportional to  $\lambda$ . The lowest such state can be represented as (for  $\mathbf{r}_0 = 0$ )

$$\psi_{\rm loc} \sim \exp\left[\frac{ixy}{2a_H^2} - \frac{x^2 + y^2}{4a_H^2}\right] \,.$$

The other levels are almost unperturbed and extended.

Now let us take into account only the lowest Landau level which we assume to be completely filled. We have one localized state and N-1 extended ones where  $N = A/2\pi a_H^2$ , which we can label by the discrete quantum number k as

$$k = \frac{p_y L_y}{2\pi\hbar} \,.$$

Each mode behaves just as the transverse mode in a quantum channel, and the current is given as

$$I = -\frac{2}{h} \sum_{nk} (E_{n,k+1} - E_{nk}) = -\frac{2}{h} \sum_{nk} (E_{n,k_{\max}} - E_{n,k_{\min}}).$$

It is not trivial to prove this equation. It was done by R. Prange using gauge considerations.

#### Proof :

The main procedure is a s follows. We have specified periodic boundary conditions along y-axis. Consider the system as a cylinder and introduce an auxiliary constant vector-potential  $\tilde{\mathbf{A}}$  along y axis as

$$\tilde{A}_y = \frac{\hbar c}{e} \frac{2\pi\alpha}{L_y}.$$
(14.2)

Increase of the parameter  $\alpha$  by one corresponds to increase the magnetic flux in the sample by 1 quantum,  $\Phi_0$ .

If one applies periodic boundary conditions then the vector-potential such as given by Eq. (14.2) can be eliminated by the gauge transform

$$\psi \to \exp(2\pi\alpha y/L_y)\psi$$

only if  $\alpha$  is integer. Thus the *extended* states which extend from 0 to  $L_y$  must depend on  $\alpha$  while localized states satisfy boundary conditions automatically since their amplitude at the boundaries vanish. The extended states in the presence of the auxiliary have the same form is in its absence, with the replacement

$$k \to k + \alpha$$
.

The vector potential A leads to an additional item in the effective Hamiltonian,

$$\delta \mathcal{H} = -\frac{1}{c} (\mathbf{I} \cdot \tilde{\mathbf{A}}).$$

Thus, the current operator can be written as

$$\hat{I}_y = -\frac{1}{c} \frac{\partial \delta \mathcal{H}}{\partial \tilde{A}_y} = -\frac{e}{h} \frac{\partial \mathcal{H}}{\partial \alpha} \,,$$

#### 14.3. EDGE CHANNELS AND ADIABATIC TRANSPORT

while the average current is

$$I = \langle \hat{I} \rangle = -\frac{e}{h} \frac{d}{d\alpha} \sum_{nk} E_{nk}(\alpha) \,.$$

As we have already mentioned, according to the construction of the quantum number k the introduction of the vector-potential leads to the replacement  $k \to k + \alpha$ . Thus

$$E_{n,k}|_{\alpha=1} = E_{n,k+1}|_{\alpha=0}$$
.

Replacing the derivative by the average value over the region  $0 \le \alpha \le 1$  we get the result given above.

Thus we come to the following picture. as the Fermi level passes the regions of the

![](_page_52_Figure_7.jpeg)

Density of states

Figure 14.3: Density of states in 2DEG in magnetic field.

extended states the steps in Hall resistance and peaks at the longitudinal resistance occur. As we have shown, the current is *independent* of the density of states, only the number of occupied extended states is important.

Now we have to remember that the state with  $k_{\max}(k_{\min})$  correspond to the upper (lower) edge of the sample if we map the quantum number k to the centers of gravity of the states. Thus we come in a natural way to edge states.

## 14.3 Edge Channels and Adiabatic Transport

The quantization of the conductance of the ballistic channel arises from the finite number of *propagating modes* each of which can carry only a very specific current. Thus it is tempting to consider the modes of an ideal finite system embedded into an external magnetic field. In

this simplified picture we can obtain some understanding concerning the nature of localized and extended states.

Let us start from a *ideal* electron system confined by the potential V(x) in the presence of the magnetic field  $\mathbf{H} \parallel \mathbf{z}$ . For a single spin component we have the Hamiltonian

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{[p_y + (eH/c)x]^2}{2m} + V(x) \,.$$

It is natural to look for a solution in the form ( $\mathcal{H}$  commutes with  $p_y$ !)

$$|n,k\rangle = \psi_{nk}(x)e^{iky}$$

where  $\hbar k$  is the eigenvalue of  $p_{y}$ . The average velocity for such a state is

$$v_n(k) = \left\langle n, k \left| \frac{p_y + (eH/c)x}{m} \right| n, k \right\rangle = \left\langle n, k \left| \frac{\partial \mathcal{H}}{\partial p_y} \right| n, k \right\rangle = \frac{1}{\hbar} \frac{dE_n(k)}{dk}.$$

It is easy to calculate such a velocity for a parabolic confinement,

$$V(x) = \frac{1}{2}m\omega_0^2 x^2 \,.$$

The result is

$$v(k) = \frac{\hbar k}{M} = \frac{\hbar k}{m} \frac{1}{1 + (\omega_c/\omega_0)^2}.$$

To understand what is going on let us consider a classical orbit with the center (X, Y). Then one can write

$$x = X + v_y/\omega_c$$
,  $y = Y - v_x/\omega_c$ .

The quantity  $r_c = v/\omega_c$  is the cyclotron radius of the orbit. We have two constants of motion, the energy E and X. In a long strip of width W the trajectories can be classified as a cyclotron orbits, skipping orbits, or traversing trajectory. In the (X, E) space such trajectories are separated by the line

$$(X \pm W/2)^2 = r_c^2$$
.

According to quantum mechanics, the motion is quantized, and we come to the following picture of quantum terms The cyclotron orbits (solid lines) correspond to Landau level, and they have zero group velocities. We have not shown traversing trajectories which correspond to higher energies. Dashed lines reproduce two sets of edge states (corresponding to skipping orbits). These sets carry the currents in opposite directions.

If the Fermi level is situated *between* the Landau levels, only the edge states can contribute to the current. Their dispersion law can be obtained (approximately) from the Bohr-Sommerfeld quantization condition,

$$h^{-1} \oint p_x \, dx + \gamma = 2\pi n \,, \quad n = 1, 2, \dots$$

![](_page_54_Figure_1.jpeg)

Figure 14.4: Typical electron trajectories in a 2D strip in magnetic field.

![](_page_54_Figure_3.jpeg)

Figure 14.5: Electron terms in a 2D strip in magnetic field.

One can show that for the rigid boundary the phase shift of the skipping orbit  $\gamma = \pi/2$ , while

$$p_x = mv_x = (eH/c)(Y-y).$$

Thus,

$$\frac{1}{a_H^2} \oint (Y - y) \, dx = 2\pi \frac{\Phi}{\Phi_0} = 2\pi \left( n - \frac{\gamma}{2\pi} \right) \, .$$

Consider an electron at the Fermi level. Its energy consists of  $(n - 1/2)\hbar\omega_c$  (+ spin contribution which I do not discuss now), as well the the part

$$E_G = \epsilon_F - (n - 1/2)\hbar\omega_c$$

due to electrostatic potential created by the edges, as well as by disorder. In an external potential, the center of the orbit moves with the drift velocity

$$v_d(\mathbf{R}) = \frac{c}{eH} \left[ \nabla V(\mathbf{R}) \times \mathbf{H} \right]$$

which is parallel to the equipotentials. That can be easily shown from classical mechanics in a magnetic field. The typical spread of the wave function along the guiding center is spread within the range of magnetic length  $a_H$ . Now we can map the classical picture to quantum mechanics according to  $\hbar k \rightarrow -x(eH/c)$ . Thus, if the typical scale of the random potential of the disorder is greater than the magnetic length, we arrive at the picture.

![](_page_55_Figure_2.jpeg)

Figure 14.6: Electron terms in the presence of long-range disorder.

Assume that the edges are in a local equilibrium. Thus if there is a difference  $\delta\zeta$  chemical potentials between the edges, then each channel contributes  $(e/h)\delta\zeta$  to the current in the Hall direction. The system appears robust because to obtain a inter-channel exchange one needs tunneling with exponentially low probability. Actually we have an almost ideal ballistic conductor and the only difference with the systems discussed earlier is that the edge channels with different directions of the current do not overlap in space.

In a typical realistic situation, the contacts are out of local equilibrium and the measured resistance depends on the properties of contacts. Consider for example, a situation when the edge channel at the lower edge are in equilibrium at chemical potential  $E_F$ , while the edge channel at the upper edge are not in local equilibrium. Then the current at the upper edge is not equipartitioned between N modes. Let  $f_n$  is the fraction of the total current I that is carried by by states above  $E_F$  in the nth channel at the upper edge,  $I_n = f_n I$ . The voltage contact at the upper edge will measure a chemical potential which depends on how it is coupled to each of the edge channels. The transmission probability  $T_n$  is the fraction of the current  $I_n$  that is transmitted through the voltage probe to a reservoir at chemical potential  $E_F + \delta \zeta$ . The incoming current

$$I_{\text{in}} = \sum_{n}^{N} T_n f_n I, \quad \text{with} \quad \sum_{n} f_n = 1, \qquad (14.3)$$

has to be balanced by an outgoing current,

$$I_{\text{out}} = \frac{e}{h} \,\delta\zeta \left(N - R\right) = \frac{e}{h} \,\delta\zeta \sum_{n} T_n \,, \tag{14.4}$$

since the voltage probe draws no net current. Thus the Hall resistance,

$$R_h = \frac{\delta\zeta}{eI} = \frac{h}{e^2} \left(\sum_n T_n f_n\right) \left(\sum_n T_n\right)^{-1}.$$
 (14.5)

The Hall conductance remains quantized only if  $f_n = 1/N$ , or at  $T_n = 1$ . The first case corresponds to local equilibrium, while the second case corresponds to an ideal contact. The Landauer-Büttiker formalism forms the basis on which anomalies in the QHE due to absence of local equilibrium in combination with non-ideal contacts can be treated theoretically.

This is a simplified picture of the integer quantum Hall effect in the random potential. The real life is much more complicated. In particular, there exists an extremely interesting *fractional* quantum Hall effect, which manifests itself at fractional values of the filling factor. We do not discuss this effect in the present course.

### Role of localization

As we have seen, at H = 0 a 2D system with disorder should have its states localized at all energies. However, only extended states are sensitive to the flux and can provide the QHE. At the same time, ranges of energies with only localized states are needed to pin  $E_F$ there and have finite plateaus. Thus, introduction of magnetic field must *delocalize some states*. As we have seen, extended modes appear near edges. However, extended states in a magnetic field are present also in the bulk of the conductor.

To discuss this phenomenon let us recall the main relevant quantities. First, let us note that the condition

$$\omega_c \tau \gg 1$$
, or  $r_c \ll \ell$ ,  $r_c = v_F/0_c$ 

for cyclotron motion is fully classical. In terms of quantum mechanical length,  $a_H = \sqrt{cH/eH}$  the classical cyclotron radius  $r_c$  can be written as

$$r_c \sim k_F a_H^2 \sim a_H \sqrt{E_F \hbar \omega_c} \sim a_H \sqrt{N}$$

where N is the number of full Landau levels. The *weak localization regime* corresponds to the inequality

$$a_H \ll \ell$$
,

while the intermediate regime where  $a_H \ll r_c$  while  $r_c$  can be comparable with  $\ell$  also exists.

#### Strong magnetic field, $\omega_c \tau \gg 1$ , $r_c \ll \ell$ .

As we have discussed, in a uniform electric field the drift velocity directed along  $[\mathbf{E} \times \mathbf{H}]$ appears,  $v_d = c(E/H)$ . This concept can be generalized for the case of a smooth random potential V(x) with does not change a lot on the scale of cyclotron motion. Then, it  $r_c$  is much less than the correlation length d of the random potential, the guiding center moves along the equipotential line  $V(\mathbf{r}) = V$ . If its orbit is closed and embeds the area A(V), then the typical frequency is

$$\frac{\omega_d}{2\pi} = \left[\oint \frac{dl}{v_d}\right]^{-1} = \frac{2c}{H} \left[\int \frac{dl \, dx_\perp}{dV}\right]^{-1} = \frac{2c}{H} \frac{\Delta V}{\Delta A}$$

where  $dx_{\perp}$  is an element of length in the direction of the potential gradient. Such a slow motion can be quantized for any Landau levels into locally equidistant levels with the separation  $\hbar 0_d$ . The area between two quantized orbits is given by the relation

$$H \Delta A = \frac{\hbar c}{e} = \Phi_0; \quad \Delta A = 2\pi a_H^2.$$

Thus the flux of H in the area per state in a given Landau level corresponds to a flux quantum, as for free electron.

Let us assume that the amplitude of the random potential is much less than  $\hbar\omega_c$ , so there is no inter-Landau-level mixing. Then the potential energy of a properly quantized levels should be added to  $\hbar\omega_c(j + 1/2)$  for *j*th Landau band. The levels correspond to the orbits running around the potential "hills", or inside potential "lakes". Both kinds of states are localized. There is one and only one energy,  $E_c$ , (in 2D case) at which the equipotential curves span the whole system (imagine filling up of the potential  $V(\mathbf{r})$ "terrain" with water). The characteristic size of the orbit,  $\xi_p$ , may be defined by the r.m.s. of the area enclosed by the equipotential contours for the localized states. It must blow up at  $E \to E_c$ ,

$$\xi_p \sim |E - E_c|^{-\nu_p}$$
,  $\nu_p \gtrsim 1$ .

Such an orbit provides the way to transfer an electron between the edges.

There is also a very interesting intermediate situation when

$$a_H \ll \ell \ll r_c$$
, or  $\omega_c \tau \ll 1$ .

As was shown by Khmelnitskii (1984), even in this region QHE plateaus can exist which are irrelevant to Landau levels.

### 14.4 Fractional Quantum Hall Effect

Fractional quantum Hall effect (FQHE) was discovered by the group of Tsui et Bell Laboratories [31]. Using a high-mobility GaAs/AlGaAs heterostructures they observed quantization of Hall conductance at filling factors  $\nu = 1/3$  and 2/3 at very low temperatures (below 1 K). Later more rich structure, as shown in Figs. 14.7 and 14.8 at fractional filling factors was discovered. It appears that only account of of Coulomb interaction leads to understanding of the problem. Now the studies of FQHE belong to the most active area of research. Below we shall provide a very brief sketch of the problems regarding electron-electron interaction in magnetic field and FQHE.

![](_page_58_Figure_1.jpeg)

The fractional quantum Hall effect. The Hall resistivity (upper curve) of the inversion layer in a high mobility AlGaAs/GaAs heterostructure shows plateaus at magnetic inductions *B* that correspond to the indicated filling factors  $\nu$ . At the same filling factors, the magnetoresistivity (lower curve) shows minima

Figure 14.7:

### Few electron with Coulomb interaction

The role of electron-electron interaction is determined by the relation between the mean free distance between electrons,  $r_s$ , and the Bohr radius,  $a_B = \epsilon \hbar^2/me^2$ . At  $r_s \ll a_B$  one can use the usual mean field description of interacting electrons, considering screening, plasmons, charge density waves, etc. However, at  $r_s \ge a_B$  the interaction energy becomes larger than the average kinetic energy. As a result, there exists a strong electron-electron correlation, and the electrons tend to crystallize. It is known that magnetic field enhances these effects.

To get some understanding let us start with more simple problem of few electrons in a magnetic field. Historically, these studies appeared important because they led to discovery of a new state, the *incompressible electron liquid*, that is believed to transform into (Wigner) crystal at very low densities.

**Two electrons.** Let us discuss the case of 2 electrons in a very strong magnetic field,  $\hbar\omega_c \geq e^2/\epsilon a_H$ . This inequality means that Landau levels are not mixed by the Coulomb interaction (see below).

Using symmetric gauge,  $\mathbf{A} = (-Hy/2, Hx/2, 0)$  and introducing polar coordinates we

![](_page_59_Figure_1.jpeg)

Figure 14.8: Recent results on the fractional quantum Hall effect.

easily obtain zeroth approximation Hamiltonian

$$\mathbf{H}_{0} = -\frac{\hbar^{2}}{2m} \nabla^{2} + \frac{m\omega_{c}^{2}\rho^{2}}{8} + \frac{\hbar\omega_{c}}{2i} \frac{\partial}{\partial\varphi} ,$$

$$\nabla^{2} = \frac{\partial^{2}}{\partial\rho^{2}} + \frac{1}{\rho} \frac{\partial}{\partial\rho} + \frac{1}{\rho^{2}} \frac{\partial^{2}}{\partial\varphi^{2}} .$$

This Hamiltonian commutes with the angular momentum,

$$l_z = \frac{\hbar}{i} \frac{\partial}{\partial \varphi} \,.$$

Thus it is natural to classify the states using the eigenvalues  $\hbar m$  of the angular momentum,  $l_z$ . The eigenfunctions have the form

$$\psi_{nm}(\rho,\varphi) = \frac{N_{nm}}{a_H^{|m|+1}} \rho^{|m|} e^{im\varphi} \exp\left(-\frac{\rho^2}{4a_H^2}\right) L_n^{|m|} \left(\frac{\rho^2}{2a_H^2}\right) \,. \tag{14.6}$$

Here *n* is non-negative integer, *m* is integer,  $L_n^{|m|}$  are Laguerre polynomials, while  $N_{nm} = \sqrt{n!/2\pi 2^{|m|}(|m|+n)}$  are normalization factors. The energy eigenvalues are

$$E_{nm} = \hbar \omega_c [n + (|m| - m + 1)/2].$$
(14.7)

The lowest Landau level corresponds to n = 0, m > 0. The Coulomb energy for the lowest state can be easily calculated as

$$E_C = \langle 0m|e^2/\epsilon r|0m\rangle = \frac{e^2}{\epsilon a_H} \frac{\Gamma(m+1/2)}{m!} \,. \tag{14.8}$$

At large m is decays as  $m^{-1/2}$ .

Two electrons are described by the Hamiltonian

$$\mathcal{H}_0(1) + \mathcal{H}_0(2) + \mathcal{H}_{int}$$

It can be rewritten through the center-of-mass coordinate,  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/\sqrt{2}$ , and the relative coordinate,  $\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)/\sqrt{2}$ , as

$$\mathcal{H}_0(\mathbf{R}) + \mathcal{H}_0(\mathbf{r}) + \mathcal{H}_{int}(\mathbf{r}\sqrt{2})$$
.

As a result, the center-of-mass motion is separated, and we look for a solution in the form

$$\Psi(\mathbf{R},\mathbf{r}) = \phi(\mathbf{R})\psi(\mathbf{r})\,.$$

Now we are left with a *single-particle* problem for the relative motion.

Since the interaction energy is radially-symmetric one can look for a solution in the form

$$\psi(\mathbf{r}) = \mathcal{R}(r)e^{-im\varphi}$$

with odd m because of the Pauli principle  $[\psi(-\mathbf{r}) = -\psi(\mathbf{r})]$ . The radial Schrödinger equation is easily written in the dimensionless units as

$$-\frac{1}{2}\frac{d^{2}\mathcal{R}}{dr^{2}} - \frac{1}{r}\frac{d\mathcal{R}}{dr} + \frac{1}{2}\left(\frac{m^{2}}{r^{2}} + m + \frac{r^{2}}{4} - \frac{\alpha}{r}\right)\mathcal{R} = E\mathcal{R}, \qquad (14.9)$$

where r is measures in units of  $a_H$ , E is measured in the units of  $\hbar\omega_c$ , while dimensionless interaction constant is  $\alpha = \sqrt{2}e^2/\epsilon a_H \hbar\omega_c$ . At large magnetic field this equation can be solved perturbatively with respect to  $\alpha$ . In the lowest approximation we obtain:

$$E_{0m}^{(1)} = \frac{\hbar\omega_c}{2} + \frac{e^2}{\epsilon a_H} \frac{\Gamma(m+1/2)}{m!} \,. \tag{14.10}$$

The energy of the center-of-mass motion must be added.

We find that the interaction *destroys* the degeneracy of the lowest Landau level. At large m the correction decreases because the electrons are less sensitive to interaction at long distances.

Three electrons. For 3 electrons we can also strip the center-of-mass motion. It can be done by the transform  $\rho = O\mathbf{r}$  where

$$\mathcal{O} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0\\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \,.$$

After the transform the interaction Hamiltonian can be written as

$$\mathcal{H}_{\rm int} = \frac{e^2}{\epsilon\sqrt{2}} \left( \frac{1}{\rho_1} + \frac{2}{|\rho_1 + \sqrt{3}\rho_2|} + \frac{2}{|\sqrt{3}\rho_2 - \rho_1|} \right) \,. \tag{14.11}$$

Again, we can write the eigen function as a product

$$\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_3) = \phi(\boldsymbol{\rho}_3)\psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$$

and in this way to reduce the problem to a two-particle one.

An important point is that the probability density must be invariant under rotation about multiples of  $\pi/3$ . The resulting Hamiltonian also commutes with the total angular momentum, **L**. Then the states can be classified according to eigenvalues M of the orbital momentum. It was shown by R. Laughlin that a proper complete set to diagonalize a 3-electron system can be written as

$$|m,m'\rangle = \frac{F}{2} \left[ (z_2 + iz_1)^{3m} - (z_2 - iz_1)^{3m} \right] (z_1^2 + z_2^2)^{m'} e^{-(|z_1|^2 + |z_2|^2)} .$$
(14.12)

Here  $z_i = \xi_i + \eta_i$ ,  $\xi$ , *eta* are the Cartesian components of the vector  $\rho_i/a_H$ , F is a normalization factor. The states (14.12) are the eigenstates of the total angular momentum with M = 3(m + m').

To diagonalize the system one has to solve the secular equation

$$\det |e\delta_{mm}\delta_{m'm'} - \langle mm'|\mathcal{H}_{\rm int}|mm'\rangle| = 0.$$

The crucial point is that the basis (14.12) is an extremely good starting approximation since off-diagonal elements of  $\mathcal{H}_{int}$  are typically at least 10 times less than the diagonal ones.

The minimum angular momentum for which a non-degenerate solution exists is M = 3 (m = 0, m' = 1). The next solution corresponds to M = 9, it is combined from the states (3,0) and (1,3). These states have the lowest energy at  $\mathcal{H}_{int} = 0$ . The "charge density" for the state with M = 9 is 1/3 comparing to the state with M = 3. Since the angular momentum is conserved and the angular momentum corresponds to the area of an electronic state, the 3 electrons are "incompressible".

#### Fractional quantum Hall states

It is impossible to diagonalize exactly the system of many electron states. An extremely effective approximate guess was suggested by R. Laughlin which we shall discuss for the case of very large magnetic field when only the lowest Landau level is important. The single-electron states for that case can be written as

$$\langle \mathbf{r}|0m\rangle = \frac{N_m}{a_H} z^m e^{-|z|^2/4} \,,$$

where z = x + iy. The complete set of N-electron states with total angular momentum  $M = \sum_{\nu=1}^{N} m_{\nu}$  are the Slater determinants

$$\Psi(1\dots N) = \sum_{P(\nu_1\dots\nu_N)} (-1)^P \prod_{\mu=1}^N N_{m_\mu} z_{\nu_\mu}^{m_\mu} \exp\left(-\frac{1}{4} \sum_{\alpha=1}^N |z_\alpha|^2\right) \,.$$

Since the ground state in the independent band approximation is a combination of Slater determinants, its general form is

$$\Psi(1\dots N) = \prod_{j < k} f(z_j - z_k) \exp\left(-\frac{1}{4} \sum_{\alpha=1}^N |z_\alpha|^2\right) \,.$$

There are several requirements to the functions f(z):

- The function f(z) must be a polynomial of z;
- Since  $\Psi$  should be a Fermion state, f(-z) = -f(z);
- $\Psi$  can be chosen as an eigenfunction of the total angular momentum. Therefore, the function f(z) has to be *homogeneous*.

The simplest choice is

$$f(z) = z^m$$
,  $(n \text{ odd})$ .

Thus the approximate wave function has the form

$$\Psi(1\dots N) = \prod_{j < k} (z_j - z_k)^m \exp\left(-\frac{1}{4}\sum_{\alpha=1}^N |z_{\alpha}|^2\right).$$
(14.13)

The Laughlin state (14.13) describes a liquid-like system. The two-particle correlation function

$$g_2^{(m)}(z_z, z_2) = \int \prod_{\nu=3}^N d\mathbf{r}_{\nu} |\Psi(1 \dots N)|^2$$

at small distances is proportional to  $|z_1 - z_2|^m$  that reflects the Pauli principle for the electrons. The smallest possible value of m is 3. The total angular momentum is just

M = Nm, while the area covered by the electrons is  $A = N(2\pi ma_H^2)$ . Thus the average electron density is  $(2\pi ma_H^2)^{-1}$ , and the filling factor is  $\nu = 1/m$ . To keep electrostatic stability one has to add the positive background.

The estimate for the Coulomb energy for the Laughlin state can me obtained as

$$E_C^{(m)} = \frac{N(N-1)}{2A} \frac{e^2}{\epsilon} \int \frac{d^2r}{r} \left[ g_2^{(m)}(r) - 1 \right] \,.$$

Because the correlation function decay strongly at small distances the incompressible liquid state appears more stable than the Wigner crystal. An interesting fact is that the Laughlin state appears the exact ground state for  $\nu = 1/m$  in the case of contact interaction,  $\mathcal{H}_{int}(\mathbf{r}) \propto \delta(\mathbf{r})$ .

As a consequence of the electron-hole symmetry it is easy to find the state corresponding to the filling factor  $(1 - \nu)$  if the state for  $\nu$  is known.

### **Elementary excitations**

Elementary excitation are important both for transport and dynamics. Changing of energy of the electron system can be achieved by its compression, or, equivalently, by changing of angular momentum while keeping the neutralizing background.

In other words, it means that new quasielectrons of quasiholes are introduced into the state  $\Psi_{\nu}$  if  $\nu \neq 1/m$ . An introduction of a quasihole can be represented as

$$\Psi_{\nu}^{+} = A^{+}(z_0)\Psi_{\nu}(z_1\dots z_n), \quad A^{+}(z_0) = \prod_{j=1}^{N} (z_j - z_0).$$

Let us estimate the effective charge of this excitation. The average area per particle which is covered in the state with filling factor  $\nu = 1/m$  is  $(N-1)(2\pi m a_H^2)$ . It can be seen by direct calculation of the integral. The corresponding charge density is

$$\rho_0 = \frac{-Ne}{(N-1)(2\pi m a_H^2)} \approx -\frac{e}{2\pi m a_H^2}.$$

Thus, each electron occupies the area with m flux quanta. Its charge must be compensated by a positive background.

In the state  $\Psi_{\nu}^{+}$  the maximum angular momentum per particle is increased by  $2\pi a_{H}^{2}$ . This corresponds to the change in the charge density which is equivalent to the positive charge +e/m.

Quasielectrons can be created in a similar way,

$$\Psi_{\nu}^{-} = A^{-}(z_0)\Psi_{\nu}(z_1\dots z_n), \quad A^{+}(z_0) = \prod_{j=1}^{N} \left(\frac{\partial}{\partial z_j} - z_0^*\right).$$

Here the partial derivative acts only on the polynomial part of the wave function  $\Psi_{\nu}$  leaving alone the Gaussian part. It can be shown that the effective charge of the quasielectron is -e/m.

The gaps between the ground and excited states were observed directly from temperature dependences of conductance. It appears that the quasiparticles can be considered as particle with so-called fractional statistics – *anyons*. Very interesting *collective excitations* were also predicted and observed experimentally by inelastic light scattering. H. Störmer, D. Tsui and R. Laughlin were awarded by the Nobel Prize 1998 for their discovery of FQHE.

However, the story is not over. Very specific features of Hall conductance were observed at  $\nu = p/q$  where p, q are integers, both for odd and even denominator q. These features were not explained by original theory by Laughlin. It appears, that at odd denominators the electrons also condense in some quantum liquids. However, the properties of that liquids differ significantly from those of the incompressible Laughlin liquid.

The above discussion is definitely not an "explanation" of FQHE. It just demonstrates some basic trends in the field. More work has to be done to understand the whole physical picture and to construct a proper transport theory.

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